Pseudo-differential operators

Summer School - From kinetic equations to statistical mechanics

Jean-Marc Bouclet
Université de Toulouse 3

Saint Jean de Monts, 28 June-2 July

Notation

Japanese bracket: $\langle\zeta\rangle := (1+|\zeta|^2)^{\frac{1}{2}}$

C_b^∞: space of smooth functions bounded together with all their derivatives

Fourier transform: $\hat{u}(\xi) = \int e^{-ix\cdot\xi}u(x)dx$

Multiindices: for $x \in \mathbb{R}^n$, $\alpha \in \mathbb{N}^n$, $x^\alpha := x_1^{\alpha_1}\cdots x_n^{\alpha_n}$,
$|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $\partial_x^\alpha = \partial_{x_1}^{\alpha}\cdots\partial_{x_n}^{\alpha_n}$

$D_x = i^{-1}\nabla_x$ and $D_x^\alpha = i^{-|\alpha|}\partial_x^\alpha$

Schwartz space: $\mathcal{S} \ni u$ if $x^\alpha\partial_x^\beta u \in L^\infty$ for all α, β

Temperate distributions: $\mathcal{S}'$

$\langle T, u\rangle$ = bilinear pairing between $T \in \mathcal{S}'$ and $u \in \mathcal{S}$, "$\int Tu$"

$(T, u) = \overline{\langle T, \bar{u}\rangle}$ = sesquilinear pairing between T and u, "$\int \bar{T}u$"

ΨDO (part 1): Symbol classes

Definition. Let $m, \mu \in \mathbb{R}$: $a \in S^{\mu,m}$ iff for all $\alpha, \beta \in \mathbb{N}^n$

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \le C_{\alpha\beta} \langle x \rangle^\mu \langle \xi \rangle^{m-|\beta|}, \qquad (x,\xi) \in \mathbb{R}^{2n}.$$

Remark. When $\mu = 0$, $S^{0,m} =: S^m$ is the most standard class.

Example. If $m \in \mathbb{N}$ and $a_\alpha \in C_b^\infty(\mathbb{R}^n)$

$$a(x,\xi) := \sum_{|\alpha| \le m} a_\alpha(x) \xi^\alpha \in S^m.$$

Facts.

- If $m \le m'$ and $\mu \le \mu'$, $S^{\mu,m}$ is contained in $S^{\mu',m'}$.
- If $a \in S^{\mu_1,m_1}$ and $b \in S^{\mu_2,m_2}$ then $ab \in S^{\mu_1+\mu_2,m_1+m_2}$.

ΨDO (part 1): Symbol classes

Definition. Let $m, \mu \in \mathbb{R}$: $a \in S^{\mu,m}$ iff for all $\alpha, \beta \in \mathbb{N}^n$

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq C_{\alpha\beta} \langle x \rangle^\mu \langle \xi \rangle^{m-|\beta|}, \qquad (x,\xi) \in \mathbb{R}^{2n}.$$

Remark. When $\mu = 0$, $S^{0,m} =: S^m$ is the most standard class.

Example. If $m \in \mathbb{N}$ and $a_\alpha \in C_b^\infty(\mathbb{R}^n)$

$$a(x,\xi) := \sum_{|\alpha| \leq m} a_\alpha(x) \xi^\alpha \in S^m.$$

Facts.

- If $m \leq m'$ and $\mu \leq \mu'$, $S^{\mu,m}$ is contained in $S^{\mu',m'}$.
- If $a \in S^{\mu_1,m_1}$ and $b \in S^{\mu_2,m_2}$ then $ab \in S^{\mu_1+\mu_2,m_1+m_2}$.

ΨDO (part 1): Symbol classes

Definition. Let $m, \mu \in \mathbb{R}$: $a \in S^{\mu,m}$ iff for all $\alpha, \beta \in \mathbb{N}^n$

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \le C_{\alpha\beta} \langle x \rangle^\mu \langle \xi \rangle^{m-|\beta|}, \qquad (x,\xi) \in \mathbb{R}^{2n}.$$

Remark. When $\mu = 0$, $S^{0,m} =: S^m$ is the most standard class.

Example. If $m \in \mathbb{N}$ and $a_\alpha \in C_b^\infty(\mathbb{R}^n)$

$$a(x,\xi) := \sum_{|\alpha| \le m} a_\alpha(x) \xi^\alpha \in S^m.$$

Facts.

- If $m \le m'$ and $\mu \le \mu'$, $S^{\mu,m}$ is contained in $S^{\mu',m'}$.
- If $a \in S^{\mu_1,m_1}$ and $b \in S^{\mu_2,m_2}$ then $ab \in S^{\mu_1+\mu_2,m_1+m_2}$.

ΨDO (part 1): Symbol classes

Definition. Let $m, \mu \in \mathbb{R}$: $a \in S^{\mu,m}$ iff for all $\alpha, \beta \in \mathbb{N}^n$

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq C_{\alpha\beta} \langle x \rangle^\mu \langle \xi \rangle^{m-|\beta|}, \qquad (x,\xi) \in \mathbb{R}^{2n}.$$

Remark. When $\mu = 0$, $S^{0,m} =: S^m$ is the most standard class.

Example. If $m \in \mathbb{N}$ and $a_\alpha \in C_b^\infty(\mathbb{R}^n)$

$$a(x,\xi) := \sum_{|\alpha| \leq m} a_\alpha(x) \xi^\alpha \in S^m.$$

Facts.

- If $m \leq m'$ and $\mu \leq \mu'$, $S^{\mu,m}$ is contained in $S^{\mu',m'}$.
- If $a \in S^{\mu_1,m_1}$ and $b \in S^{\mu_2,m_2}$ then $ab \in S^{\mu_1+\mu_2,m_1+m_2}$.

ΨDO (part 1): Symbol classes

Definition. Let $m, \mu \in \mathbb{R}$: $a \in S^{\mu,m}$ iff for all $\alpha, \beta \in \mathbb{N}^n$

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq C_{\alpha\beta} \langle x \rangle^\mu \langle \xi \rangle^{m-|\beta|}, \qquad (x,\xi) \in \mathbb{R}^{2n}.$$

Remark. When $\mu = 0$, $S^{0,m} =: S^m$ is the most standard class.

Example. If $m \in \mathbb{N}$ and $a_\alpha \in C_b^\infty(\mathbb{R}^n)$

$$a(x,\xi) := \sum_{|\alpha| \leq m} a_\alpha(x) \xi^\alpha \in S^m.$$

Facts.

- If $m \leq m'$ and $\mu \leq \mu'$, $S^{\mu,m}$ is contained in $S^{\mu',m'}$.
- If $a \in S^{\mu_1,m_1}$ and $b \in S^{\mu_2,m_2}$ then $ab \in S^{\mu_1+\mu_2,m_1+m_2}$.

ΨDO (part 1): Symbol classes

Definition. Let $m, \mu \in \mathbb{R}$: $a \in S^{\mu,m}$ iff for all $\alpha, \beta \in \mathbb{N}^n$

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \le C_{\alpha\beta} \langle x \rangle^\mu \langle \xi \rangle^{m-|\beta|}, \qquad (x,\xi) \in \mathbb{R}^{2n}.$$

Remark. When $\mu = 0$, $S^{0,m} =: S^m$ is the most standard class.

Example. If $m \in \mathbb{N}$ and $a_\alpha \in C_b^\infty(\mathbb{R}^n)$

$$a(x,\xi) := \sum_{|\alpha| \le m} a_\alpha(x) \xi^\alpha \in S^m.$$

Facts.

- If $m \le m'$ and $\mu \le \mu'$, $S^{\mu,m}$ is contained in $S^{\mu',m'}$.
- If $a \in S^{\mu_1,m_1}$ and $b \in S^{\mu_2,m_2}$ then $ab \in S^{\mu_1+\mu_2,m_1+m_2}$.

ΨDO (part 1): Symbol classes

Definition. Let $m, \mu \in \mathbb{R}$: $a \in S^{\mu,m}$ iff for all $\alpha, \beta \in \mathbb{N}^n$

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq C_{\alpha\beta} \langle x \rangle^\mu \langle \xi \rangle^{m-|\beta|}, \qquad (x,\xi) \in \mathbb{R}^{2n}.$$

Remark. When $\mu = 0$, $S^{0,m} =: S^m$ is the most standard class.

Example. If $m \in \mathbb{N}$ and $a_\alpha \in C_b^\infty(\mathbb{R}^n)$

$$a(x,\xi) := \sum_{|\alpha| \leq m} a_\alpha(x) \xi^\alpha \in S^m.$$

Facts.

- If $m \leq m'$ and $\mu \leq \mu'$, $S^{\mu,m}$ is contained in $S^{\mu',m'}$.
- If $a \in S^{\mu_1,m_1}$ and $b \in S^{\mu_2,m_2}$ then $ab \in S^{\mu_1+\mu_2,m_1+m_2}$.

ΨDO (part 1): Symbol classes

Topology of $S^{\mu,m}$. Fréchet space for the sequence of (semi)norms

$$||a||_{S^{\mu,m},N} := \max_{|\alpha+\beta|\le N} \sup_{\mathbb{R}^{2n}} \left| \langle x\rangle^{-\mu}\langle\xi\rangle^{-m+|\beta|}\partial_x^\alpha\partial_\xi^\beta a(x,\xi)\right|.$$

Remark. A linear (or conjugate linear) map B from S^{μ_1,m_1} to S^{μ_2,m_2} is continuous if: $\forall\, N_2\in\mathbb{N}\ \ \exists C>0$ and $\exists N_1\in\mathbb{N}$ s.t.

$$||B(a)||_{S^{\mu_2,m_2},N_2} \le C||a||_{S^{\mu_1,m_1},N_1}, \qquad a\in S^{\mu_1,m_1}.$$

Proposition.

- If $m\le m'$, $\mu\le\mu'$, $S^{\mu,m}\subset S^{\mu',m'}$ continuously
- $S^{\mu_1,m_1}\times S^{\mu_2,m_2}\ni(a,b)\mapsto ab\in S^{\mu_1+\mu_2,m_1+m_2}$ is continuous
- If $\chi\in C_0^\infty$, $\chi\equiv 1$ near 0 and $a_\epsilon(x,\xi):=\chi(\epsilon x,\epsilon\xi)a(x,\xi)$ with $a\in S^{\mu,m}$, $a_\epsilon\to a$ in $S^{\mu',m'}$ as $\epsilon\to 0\ \forall m'>m$, $\mu'>\mu$. Furthermore $(a_\epsilon)_{\epsilon\in(0,1]}$ is bounded in $S^{\mu,m}$.

ΨDO (part 1): Symbol classes

Topology of $S^{\mu,m}$. Fréchet space for the sequence of (semi)norms

$$||a||_{S^{\mu,m},N} := \max_{|\alpha+\beta|\le N} \sup_{\mathbb{R}^{2n}} \left| \langle x\rangle^{-\mu} \langle \xi\rangle^{-m+|\beta|} \partial_x^\alpha \partial_\xi^\beta a(x,\xi) \right| .$$

Remark. A linear (or conjugate linear) map B from S^{μ_1,m_1} to S^{μ_2,m_2} is continuous if: $\forall\ N_2 \in \mathbb{N}\ \ \exists C > 0$ and $\exists N_1 \in \mathbb{N}$ s.t.

$$||B(a)||_{S^{\mu_2,m_2},N_2} \le C||a||_{S^{\mu_1,m_1},N_1}, \qquad a \in S^{\mu_1,m_1}.$$

Proposition.

- If $m \le m'$, $\mu \le \mu'$, $S^{\mu,m} \subset S^{\mu',m'}$ continuously
- $S^{\mu_1,m_1} \times S^{\mu_2,m_2} \ni (a,b) \mapsto ab \in S^{\mu_1+\mu_2,m_1+m_2}$ is continuous
- If $\chi \in C_0^\infty$, $\chi \equiv 1$ near 0 and $a_\epsilon(x,\xi) := \chi(\epsilon x, \epsilon\xi)a(x,\xi)$ with $a \in S^{\mu,m}$, $a_\epsilon \to a$ in $S^{\mu',m'}$ as $\epsilon \to 0\ \forall m' > m$, $\mu' > \mu$. Furthermore $(a_\epsilon)_{\epsilon\in(0,1]}$ is bounded in $S^{\mu,m}$.

ΨDO (part 1): Symbol classes

Topology of $S^{\mu,m}$. Fréchet space for the sequence of (semi)norms

$$||a||_{S^{\mu,m},N} := \max_{|\alpha+\beta|\leq N} \sup_{\mathbb{R}^{2n}} \left| \langle x\rangle^{-\mu} \langle \xi\rangle^{-m+|\beta|} \partial_x^\alpha \partial_\xi^\beta a(x,\xi) \right| .$$

Remark. A linear (or conjugate linear) map B from S^{μ_1,m_1} to S^{μ_2,m_2} is continuous if: $\forall\ N_2 \in \mathbb{N}\ \ \exists C > 0$ and $\exists N_1 \in \mathbb{N}$ s.t.

$$||B(a)||_{S^{\mu_2,m_2},N_2} \leq C||a||_{S^{\mu_1,m_1},N_1}, \qquad a \in S^{\mu_1,m_1}.$$

Proposition.

- If $m \leq m'$, $\mu \leq \mu'$, $S^{\mu,m} \subset S^{\mu',m'}$ continuously
- $S^{\mu_1,m_1} \times S^{\mu_2,m_2} \ni (a,b) \mapsto ab \in S^{\mu_1+\mu_2,m_1+m_2}$ is continuous
- If $\chi \in C_0^\infty$, $\chi \equiv 1$ near 0 and $a_\epsilon(x,\xi) := \chi(\epsilon x, \epsilon\xi)a(x,\xi)$ with $a \in S^{\mu,m}$, $a_\epsilon \to a$ in $S^{\mu',m'}$ as $\epsilon \to 0$ $\forall m' > m$, $\mu' > \mu$. Furthermore $(a_\epsilon)_{\epsilon\in(0,1]}$ is bounded in $S^{\mu,m}$.

ΨDO (part 1): Symbol classes

Topology of $S^{\mu,m}$. Fréchet space for the sequence of (semi)norms

$$||a||_{S^{\mu,m},N} := \max_{|\alpha+\beta|\leq N} \sup_{\mathbb{R}^{2n}} \left| \langle x \rangle^{-\mu} \langle \xi \rangle^{-m+|\beta|} \partial_x^\alpha \partial_\xi^\beta a(x,\xi) \right|.$$

Remark. A linear (or conjugate linear) map B from S^{μ_1,m_1} to S^{μ_2,m_2} is continuous if: $\forall\, N_2 \in \mathbb{N}\ \ \exists C > 0$ and $\exists N_1 \in \mathbb{N}$ s.t.

$$||B(a)||_{S^{\mu_2,m_2},N_2} \leq C||a||_{S^{\mu_1,m_1},N_1}, \qquad a \in S^{\mu_1,m_1}.$$

Proposition.

- If $m \leq m'$, $\mu \leq \mu'$, $S^{\mu,m} \subset S^{\mu',m'}$ continuously
- $S^{\mu_1,m_1} \times S^{\mu_2,m_2} \ni (a,b) \mapsto ab \in S^{\mu_1+\mu_2,m_1+m_2}$ is continuous
- If $\chi \in C_0^\infty$, $\chi \equiv 1$ near 0 and $a_\epsilon(x,\xi) := \chi(\epsilon x, \epsilon\xi)a(x,\xi)$ with $a \in S^{\mu,m}$, $a_\epsilon \to a$ in $S^{\mu',m'}$ as $\epsilon \to 0$ $\forall m' > m$, $\mu' > \mu$. Furthermore $(a_\epsilon)_{\epsilon\in(0,1]}$ is bounded in $S^{\mu,m}$.

ΨDO (part 1): Symbol classes

Topology of $S^{\mu,m}$. Fréchet space for the sequence of (semi)norms

$$||a||_{S^{\mu,m},N} := \max_{|\alpha+\beta|\leq N} \sup_{\mathbb{R}^{2n}} \left| \langle x \rangle^{-\mu} \langle \xi \rangle^{-m+|\beta|} \partial_x^\alpha \partial_\xi^\beta a(x,\xi) \right| .$$

Remark. A linear (or conjugate linear) map B from S^{μ_1,m_1} to S^{μ_2,m_2} is continuous if: $\forall\ N_2 \in \mathbb{N}\ \ \exists C > 0$ and $\exists N_1 \in \mathbb{N}$ s.t.

$$||B(a)||_{S^{\mu_2,m_2},N_2} \leq C||a||_{S^{\mu_1,m_1},N_1}, \qquad a \in S^{\mu_1,m_1}.$$

Proposition.

- If $m \leq m'$, $\mu \leq \mu'$, $S^{\mu,m} \subset S^{\mu',m'}$ continuously
- $S^{\mu_1,m_1} \times S^{\mu_2,m_2} \ni (a,b) \mapsto ab \in S^{\mu_1+\mu_2,m_1+m_2}$ is continuous
- If $\chi \in C_0^\infty$, $\chi \equiv 1$ near 0 and $a_\epsilon(x,\xi) := \chi(\epsilon x, \epsilon\xi)a(x,\xi)$ with $a \in S^{\mu,m}$, $a_\epsilon \to a$ in $S^{\mu',m'}$ as $\epsilon \to 0$ $\forall m' > m$, $\mu' > \mu$. Furthermore $(a_\epsilon)_{\epsilon\in(0,1]}$ is bounded in $S^{\mu,m}$.

ΨDO (part 1): Symbol classes

Topology of $S^{\mu,m}$. Fréchet space for the sequence of (semi)norms

$$||a||_{S^{\mu,m},N} := \max_{|\alpha+\beta|\le N} \sup_{\mathbb{R}^{2n}} \left| \langle x\rangle^{-\mu}\langle\xi\rangle^{-m+|\beta|}\partial_x^\alpha\partial_\xi^\beta a(x,\xi)\right|.$$

Remark. A linear (or conjugate linear) map B from S^{μ_1,m_1} to S^{μ_2,m_2} is continuous if: $\forall\ N_2\in\mathbb{N}\ \ \exists C>0$ and $\exists N_1\in\mathbb{N}$ s.t.

$$||B(a)||_{S^{\mu_2,m_2},N_2} \le C||a||_{S^{\mu_1,m_1},N_1}, \qquad a\in S^{\mu_1,m_1}.$$

Proposition.

- If $m\le m'$, $\mu\le\mu'$, $S^{\mu,m}\subset S^{\mu',m'}$ continuously
- $S^{\mu_1,m_1}\times S^{\mu_2,m_2}\ni(a,b)\mapsto ab\in S^{\mu_1+\mu_2,m_1+m_2}$ is continuous
- If $\chi\in C_0^\infty$, $\chi\equiv 1$ near 0 and $a_\epsilon(x,\xi):=\chi(\epsilon x,\epsilon\xi)a(x,\xi)$ with $a\in S^{\mu,m}$, $a_\epsilon\to a$ in $S^{\mu',m'}$ as $\epsilon\to 0\ \forall m'>m$, $\mu'>\mu$. Furthermore $(a_\epsilon)_{\epsilon\in(0,1]}$ is bounded in $S^{\mu,m}$.

ΨDO (part 1): Symbol classes

Topology of $S^{\mu,m}$. Fréchet space for the sequence of (semi)norms

$$||a||_{S^{\mu,m},N} := \max_{|\alpha+\beta|\leq N} \sup_{\mathbb{R}^{2n}} \left| \langle x\rangle^{-\mu}\langle\xi\rangle^{-m+|\beta|}\partial_x^\alpha\partial_\xi^\beta a(x,\xi)\right|.$$

Remark. A linear (or conjugate linear) map B from S^{μ_1,m_1} to S^{μ_2,m_2} is continuous if: $\forall\ N_2 \in \mathbb{N}\ \ \exists C > 0$ and $\exists N_1 \in \mathbb{N}$ s.t.

$$||B(a)||_{S^{\mu_2,m_2},N_2} \leq C||a||_{S^{\mu_1,m_1},N_1}, \qquad a \in S^{\mu_1,m_1}.$$

Proposition.

- If $m \leq m'$, $\mu \leq \mu'$, $S^{\mu,m} \subset S^{\mu',m'}$ continuously
- $S^{\mu_1,m_1} \times S^{\mu_2,m_2} \ni (a,b) \mapsto ab \in S^{\mu_1+\mu_2,m_1+m_2}$ is continuous
- If $\chi \in C_0^\infty$, $\chi \equiv 1$ near 0 and $a_\epsilon(x,\xi) := \chi(\epsilon x,\epsilon\xi)a(x,\xi)$ with $a \in S^{\mu,m}$, $a_\epsilon \to a$ in $S^{\mu',m'}$ as $\epsilon \to 0\ \forall m' > m$, $\mu' > \mu$. Furthermore $(a_\epsilon)_{\epsilon\in(0,1]}$ is bounded in $S^{\mu,m}$.

ΨDO (part 1): Definition

Definition. If $a \in S^{\mu,m}$, $u \in \mathcal{S}$ and $h \in (0,1]$,

$$a(x,hD)u(x) = Op_h(a)u(x) := (2\pi)^{-n} \int e^{ix\cdot\xi} a(x,h\xi)\hat{u}(\xi)d\xi.$$

Remark. The case $h = 1$ is relevant.

Where does this definition come from ?

More generally

$$a(x)D^\alpha u(x) = (2\pi)^{-n} \int e^{ix\cdot\xi} a(x)\xi^\alpha \hat{u}(\xi)d\xi$$

so if $p(x,\xi) := \sum_{|\alpha|\leq m} a_\alpha(x)\xi^\alpha$,

$$p(x,D) = \sum_{|\alpha|\leq m} a_\alpha(x)D^\alpha$$

Pseudo-differential operators extend this formula to a larger class of symbols p.

ΨDO (part 1): Definition

Definition. If $a \in S^{\mu,m}$, $u \in \mathcal{S}$ and $h \in (0,1]$,

$$a(x, hD)u(x) = Op_h(a)u(x) := (2\pi)^{-n} \int e^{ix\cdot\xi} a(x, h\xi)\hat{u}(\xi)d\xi.$$

Remark. The case $h = 1$ is relevant.

Where does this definition come from ?

More generally

$$a(x)D^\alpha u(x) = (2\pi)^{-n} \int e^{ix\cdot\xi} a(x)\xi^\alpha \hat{u}(\xi)d\xi$$

so if $p(x,\xi) := \sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha$,

$$p(x, D) = \sum_{|\alpha|\le m} a_\alpha(x)D^\alpha$$

Pseudo-differential operators extend this formula to a larger class of symbols p.

ΨDO (part 1): Definition

Definition. If $a \in S^{\mu,m}$, $u \in \mathcal{S}$ and $h \in (0,1]$,

$$a(x,hD)u(x) = Op_h(a)u(x) := (2\pi)^{-n} \int e^{ix\cdot\xi} a(x, h\xi)\hat{u}(\xi)d\xi.$$

Remark. The case $h = 1$ is relevant.

Where does this definition come from ?

More generally

$$a(x)D^\alpha u(x) = (2\pi)^{-n} \int e^{ix\cdot\xi} a(x)\xi^\alpha \hat{u}(\xi)d\xi$$

so if $p(x,\xi) := \sum_{|\alpha|\leq m} a_\alpha(x)\xi^\alpha$,

$$p(x,D) = \sum_{|\alpha|\leq m} a_\alpha(x)D^\alpha$$

Pseudo-differential operators extend this formula to a larger class of symbols p.

ΨDO (part 1): Definition

Definition. If $a \in S^{\mu,m}$, $u \in \mathcal{S}$ and $h \in (0,1]$,

$$a(x,hD)u(x) = Op_h(a)u(x) := (2\pi)^{-n} \int e^{ix\cdot\xi} a(x,h\xi)\hat{u}(\xi)d\xi.$$

Remark. The case $h = 1$ is relevant.

Where does this definition come from ?

More generally

$$a(x)D^\alpha u(x) = (2\pi)^{-n} \int e^{ix\cdot\xi} a(x)\xi^\alpha \hat{u}(\xi)d\xi$$

so if $p(x,\xi) := \sum_{|\alpha|\leq m} a_\alpha(x)\xi^\alpha$,

$$p(x,D) = \sum_{|\alpha|\leq m} a_\alpha(x)D^\alpha$$

Pseudo-differential operators extend this formula to a larger class of symbols p.

ΨDO (part 1): Definition

Definition. If $a \in S^{\mu,m}$, $u \in \mathcal{S}$ and $h \in (0,1]$,

$$a(x,hD)u(x) = Op_h(a)u(x) := (2\pi)^{-n} \int e^{ix\cdot\xi} a(x,h\xi)\hat{u}(\xi)d\xi.$$

Remark. The case $h = 1$ is relevant.

Where does this definition come from ?

$$u(x) = (2\pi)^{-n} \int e^{ix\cdot\xi} \hat{u}(\xi)d\xi, \qquad D_j = \frac{\partial}{i\partial x_j}$$

More generally

$$a(x)D^\alpha u(x) = (2\pi)^{-n} \int e^{ix\cdot\xi} a(x)\xi^\alpha \hat{u}(\xi)d\xi$$

so if $p(x,\xi) := \sum_{|\alpha|\leq m} a_\alpha(x)\xi^\alpha$,

$$p(x,D) = \sum_{|\alpha|\leq m} a_\alpha(x)D^\alpha$$

Pseudo-differential operators extend this formula to a larger class of symbols p.

ΨDO (part 1): Definition

Definition. If $a \in S^{\mu,m}$, $u \in \mathcal{S}$ and $h \in (0,1]$,

$$a(x, hD)u(x) = Op_h(a)u(x) := (2\pi)^{-n} \int e^{ix\cdot\xi} a(x, h\xi)\hat{u}(\xi)d\xi.$$

Remark. The case $h = 1$ is relevant.

Where does this definition come from ?

$$D_j u(x) = (2\pi)^{-n} D_j \int e^{ix\cdot\xi}\hat{u}(\xi)d\xi, \qquad D_j = \frac{\partial}{i\partial x_j}$$

More generally

$$a(x)D^\alpha u(x) = (2\pi)^{-n} \int e^{ix\cdot\xi} a(x)\xi^\alpha \hat{u}(\xi)d\xi$$

so if $p(x,\xi) := \sum_{|\alpha|\leq m} a_\alpha(x)\xi^\alpha$,

$$p(x, D) = \sum_{|\alpha|\leq m} a_\alpha(x)D^\alpha$$

Pseudo-differential operators extend this formula to a larger class of symbols p.

ΨDO (part 1): Definition

Definition. If $a \in S^{\mu,m}$, $u \in \mathcal{S}$ and $h \in (0,1]$,

$$a(x, hD)u(x) = Op_h(a)u(x) := (2\pi)^{-n} \int e^{ix\cdot\xi} a(x, h\xi)\hat{u}(\xi) d\xi.$$

Remark. The case $h = 1$ is relevant.

Where does this definition come from ?

$$D_j u(x) = (2\pi)^{-n} \int D_j e^{ix\cdot\xi} \hat{u}(\xi) d\xi, \qquad D_j = \frac{\partial}{i\partial x_j}$$

More generally

$$a(x)D^\alpha u(x) = (2\pi)^{-n} \int e^{ix\cdot\xi} a(x)\xi^\alpha \hat{u}(\xi) d\xi$$

so if $p(x,\xi) := \sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha$,

$$p(x, D) = \sum_{|\alpha|\le m} a_\alpha(x) D^\alpha$$

Pseudo-differential operators extend this formula to a larger class of symbols p.

ΨDO (part 1): Definition

Definition. If $a \in S^{\mu,m}$, $u \in \mathcal{S}$ and $h \in (0,1]$,

$$a(x, hD)u(x) = Op_h(a)u(x) := (2\pi)^{-n} \int e^{ix\cdot\xi} a(x, h\xi)\hat{u}(\xi)d\xi.$$

Remark. The case $h = 1$ is relevant.

Where does this definition come from ?

$$D_j u(x) = (2\pi)^{-n} \int \xi_j e^{ix\cdot\xi} \hat{u}(\xi)d\xi, \qquad D_j = \frac{\partial}{i\partial x_j}$$

More generally

$$a(x)D^\alpha u(x) = (2\pi)^{-n} \int e^{ix\cdot\xi} a(x)\xi^\alpha \hat{u}(\xi)d\xi$$

so if $p(x,\xi) := \sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha$,

$$p(x, D) = \sum_{|\alpha|\le m} a_\alpha(x)D^\alpha$$

Pseudo-differential operators extend this formula to a larger class of symbols p.

ΨDO (part 1): Definition

Definition. If $a \in S^{\mu,m}$, $u \in \mathcal{S}$ and $h \in (0,1]$,

$$a(x,hD)u(x) = Op_h(a)u(x) := (2\pi)^{-n} \int e^{ix\cdot\xi} a(x,h\xi)\hat{u}(\xi)d\xi.$$

Remark. The case $h = 1$ is relevant.

Where does this definition come from ?

$$D_j u(x) = (2\pi)^{-n} \int e^{ix\cdot\xi} \xi_j \hat{u}(\xi)d\xi, \qquad D_j = \frac{\partial}{i\partial x_j}$$

More generally

$$a(x)D^\alpha u(x) = (2\pi)^{-n} \int e^{ix\cdot\xi} a(x)\xi^\alpha \hat{u}(\xi)d\xi$$

so if $p(x,\xi) := \sum_{|\alpha|\leq m} a_\alpha(x)\xi^\alpha$,

$$p(x,D) = \sum_{|\alpha|\leq m} a_\alpha(x)D^\alpha$$

Pseudo-differential operators extend this formula to a larger class of symbols p.

ΨDO (part 1): Definition

Definition. If $a \in S^{\mu,m}$, $u \in \mathcal{S}$ and $h \in (0,1]$,

$$a(x, hD)u(x) = Op_h(a)u(x) := (2\pi)^{-n} \int e^{ix\cdot\xi} a(x, h\xi)\hat{u}(\xi)d\xi.$$

Remark. The case $h = 1$ is relevant.

Where does this definition come from ?

$$D_j u(x) = (2\pi)^{-n} \int e^{ix\cdot\xi} \xi_j \hat{u}(\xi)d\xi, \qquad D_j = \frac{\partial}{i\partial x_j}$$

More generally

$$a(x)D^\alpha u(x) = (2\pi)^{-n} \int e^{ix\cdot\xi} a(x)\xi^\alpha \hat{u}(\xi)d\xi$$

so if $p(x,\xi) := \sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha$,

$$p(x, D) = \sum_{|\alpha|\le m} a_\alpha(x)D^\alpha$$

Pseudo-differential operators extend this formula to a larger class of symbols p.

ΨDO (part 1): Definition

Definition. If $a \in S^{\mu,m}$, $u \in \mathcal{S}$ and $h \in (0,1]$,

$$a(x,hD)u(x) = Op_h(a)u(x) := (2\pi)^{-n} \int e^{ix\cdot\xi} a(x,h\xi)\hat{u}(\xi)d\xi.$$

Remark. The case $h = 1$ is relevant.

Where does this definition come from ?

$$D_j u(x) = (2\pi)^{-n} \int e^{ix\cdot\xi} \xi_j \hat{u}(\xi)d\xi, \qquad D_j = \frac{\partial}{i\partial x_j}$$

More generally

$$a(x)D^\alpha u(x) = (2\pi)^{-n} \int e^{ix\cdot\xi} a(x)\xi^\alpha \hat{u}(\xi)d\xi$$

so if $p(x,\xi) := \sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha$,

$$p(x,D) = \sum_{|\alpha|\le m} a_\alpha(x)D^\alpha$$

Pseudo-differential operators extend this formula to a larger class of symbols p.

ΨDO (part 1): Definition

Definition. If $a \in S^{\mu,m}$, $u \in \mathcal{S}$ and $h \in (0,1]$,

$$a(x,hD)u(x) = Op_h(a)u(x) := (2\pi)^{-n} \int e^{ix\cdot\xi} a(x, h\xi)\hat{u}(\xi)d\xi.$$

Remark. The case $h = 1$ is relevant.

Where does this definition come from ?

$$D_j u(x) = (2\pi)^{-n} \int e^{ix\cdot\xi} \xi_j \hat{u}(\xi)d\xi, \qquad D_j = \frac{\partial}{i\partial x_j}$$

More generally

$$a(x)D^\alpha u(x) = (2\pi)^{-n} \int e^{ix\cdot\xi} a(x)\xi^\alpha \hat{u}(\xi)d\xi$$

so if $p(x,\xi) := \sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha$,

$$p(x,D) = \sum_{|\alpha|\le m} a_\alpha(x)D^\alpha$$

Pseudo-differential operators extend this formula to a larger class of symbols p.

ΨDO (part 1): Definition

Definition. If $a \in S^{\mu,m}$, $u \in \mathcal{S}$ and $h \in (0,1]$,

$$a(x,hD)u(x) = Op_h(a)u(x) := (2\pi)^{-n} \int e^{ix\cdot\xi} a(x, h\xi)\hat{u}(\xi)d\xi.$$

Remark. The case $h = 1$ is relevant.

Where does this definition come from ?

$$D_j u(x) = (2\pi)^{-n} \int e^{ix\cdot\xi} \xi_j \hat{u}(\xi)d\xi, \qquad D_j = \frac{\partial}{i\partial x_j}$$

More generally

$$a(x)D^\alpha u(x) = (2\pi)^{-n} \int e^{ix\cdot\xi} a(x)\xi^\alpha \hat{u}(\xi)d\xi$$

so if $p(x,\xi) := \sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha$,

$$p(x,D) = \sum_{|\alpha|\le m} a_\alpha(x)D^\alpha$$

Pseudo-differential operators extend this formula to a larger class of symbols p.

ΨDO (part 1): Where can the *h* come from ?

- Semiclassical operators, e.g.

$$H(\hbar) = -\hbar^2\Delta + V(x) = Op_\hbar(|\xi|^2 + V(x))$$

- Large spectral parameter problem:

$$\begin{aligned} -\Delta + V(x) - \lambda e^{i\theta} &= \lambda(-\lambda^{-1}\Delta + \lambda^{-1}V(x) - e^{i\theta}) \\ &= \lambda\big(Op_h(|\xi|^2 + h^2V(x) - e^{i\theta})\big), \quad h = \frac{1}{\sqrt{\lambda}} \end{aligned}$$

- Regularity problems: if $\chi \in C_0^\infty(\mathbb{R}^n \setminus 0)$, $h = 2^{-k}$ is a *small* dyadic (Littlewood-Paley) parameter and $a \in S^m$,

$$a(x,D)\chi(2^{-k}D) = Op_1(a(x,\xi)\chi(h\xi)) = h^{-m}Op_h(\tilde{a}_h(x,\xi))$$

with

$$\tilde{a}_h(x,\xi) = h^m a(x,\xi/h)\chi(\xi) \quad \rightarrow \text{bounded family of } S^0.$$

- Short time heat kernel asymptotics: $e^{t\Delta_g}$ as $t \rightarrow 0$, $h = t^{1/2}$

ΨDO (part 1): Where can the *h* come from ?

- Semiclassical operators, e.g.

$$H(\hbar) = -\hbar^2\Delta + V(x) = Op_\hbar(|\xi|^2 + V(x))$$

- Large spectral parameter problem:

$$\begin{aligned} -\Delta + V(x) - \lambda e^{i\theta} &= \lambda(-\lambda^{-1}\Delta + \lambda^{-1}V(x) - e^{i\theta}) \\ &= \lambda\big(Op_h(|\xi|^2 + h^2 V(x) - e^{i\theta})\big), \quad h = \frac{1}{\sqrt{\lambda}} \end{aligned}$$

- Regularity problems: if $\chi \in C_0^\infty(\mathbb{R}^n \setminus 0)$, $h = 2^{-k}$ is a *small* dyadic (Littlewood-Paley) parameter and $a \in S^m$,

$$a(x, D)\chi(2^{-k}D) = Op_1(a(x,\xi)\chi(h\xi)) = h^{-m}Op_h(\tilde{a}_h(x,\xi))$$

with

$$\tilde{a}_h(x,\xi) = h^m a(x, \xi/h)\chi(\xi) \quad \to \text{bounded family of } S^0.$$

- Short time heat kernel asymptotics: $e^{t\Delta_g}$ as $t \to 0$, $h = t^{1/2}$

ΨDO (part 1): Where can the h come from ?

- Semiclassical operators, e.g.

$$H(\hbar) = -\hbar^2\Delta + V(x) = Op_\hbar(|\xi|^2 + V(x))$$

- Large spectral parameter problem:

$$\begin{aligned} -\Delta + V(x) - \lambda e^{i\theta} &= \lambda(-\lambda^{-1}\Delta + \lambda^{-1}V(x) - e^{i\theta}) \\ &= \lambda\big(Op_h(|\xi|^2 + h^2V(x) - e^{i\theta})\big), \quad h = \frac{1}{\sqrt{\lambda}} \end{aligned}$$

- Regularity problems: if $\chi \in C_0^\infty(\mathbb{R}^n \setminus 0)$, $h = 2^{-k}$ is a *small* dyadic (Littlewood-Paley) parameter and $a \in S^m$,

$$a(x,D)\chi(2^{-k}D) = Op_1(a(x,\xi)\chi(h\xi)) = h^{-m}Op_h(\tilde{a}_h(x,\xi))$$

with

$$\tilde{a}_h(x,\xi) = h^m a(x,\xi/h)\chi(\xi) \quad \to \text{bounded family of } S^0.$$

- Short time heat kernel asymptotics: $e^{t\Delta_g}$ as $t \to 0$, $h = t^{1/2}$

ΨDO (part 1): Where can the *h* come from ?

- Semiclassical operators, e.g.

$$H(\hbar) = -\hbar^2\Delta + V(x) = Op_\hbar(|\xi|^2 + V(x))$$

- Large spectral parameter problem:

$$\begin{aligned} -\Delta + V(x) - \lambda e^{i\theta} &= \lambda(-\lambda^{-1}\Delta + \lambda^{-1}V(x) - e^{i\theta}) \\ &= \lambda\big(Op_h(|\xi|^2 + h^2V(x) - e^{i\theta})\big), \quad h = \frac{1}{\sqrt{\lambda}} \end{aligned}$$

- Regularity problems: if $\chi \in C_0^\infty(\mathbb{R}^n \setminus 0)$, $h = 2^{-k}$ is a *small* dyadic (Littlewood-Paley) parameter and $a \in S^m$,

$$a(x,D)\chi(2^{-k}D) = Op_1(a(x,\xi)\chi(h\xi)) = h^{-m}Op_h(\tilde{a}_h(x,\xi))$$

with

$$\tilde{a}_h(x,\xi) = h^m a(x,\xi/h)\chi(\xi) \quad \to \text{bounded family of } S^0.$$

- Short time heat kernel asymptotics: $e^{t\Delta_g}$ as $t \to 0$, $h = t^{1/2}$

ΨDO (part 1): Where can the *h* come from ?

- Semiclassical operators, e.g.

$$H(\hbar) = -\hbar^2\Delta + V(x) = Op_\hbar(|\xi|^2 + V(x))$$

- Large spectral parameter problem:

$$\begin{aligned} -\Delta + V(x) - \lambda e^{i\theta} &= \lambda(-\lambda^{-1}\Delta + \lambda^{-1}V(x) - e^{i\theta}) \\ &= \lambda\big(Op_h(|\xi|^2 + h^2V(x) - e^{i\theta})\big), \quad h = \frac{1}{\sqrt{\lambda}} \end{aligned}$$

- Regularity problems: if $\chi \in C_0^\infty(\mathbb{R}^n \setminus 0)$, $h = 2^{-k}$ is a *small* dyadic (Littlewood-Paley) parameter and $a \in S^m$,

$$a(x,D)\chi(2^{-k}D) = Op_1(a(x,\xi)\chi(h\xi)) = h^{-m}Op_h(\tilde{a}_h(x,\xi))$$

with

$$\tilde{a}_h(x,\xi) = h^m a(x,\xi/h)\chi(\xi) \quad \to \text{bounded family of } S^0.$$

- Short time heat kernel asymptotics: $e^{t\Delta_g}$ as $t \to 0$, $h = t^{1/2}$

ΨDO (part 1): Where can the *h* come from ?

- Semiclassical operators, e.g.

$$H(\hbar) = -\hbar^2\Delta + V(x) = Op_\hbar(|\xi|^2 + V(x))$$

- Large spectral parameter problem:

$$\begin{aligned} -\Delta + V(x) - \lambda e^{i\theta} &= \lambda(-\lambda^{-1}\Delta + \lambda^{-1}V(x) - e^{i\theta}) \\ &= \lambda\big(Op_h(|\xi|^2 + h^2V(x) - e^{i\theta})\big), \quad h = \frac{1}{\sqrt{\lambda}} \end{aligned}$$

- Regularity problems: if $\chi \in C_0^\infty(\mathbb{R}^n \setminus 0)$, $h = 2^{-k}$ is a *small* dyadic (Littlewood-Paley) parameter and $a \in S^m$,

$$a(x,D)\chi(2^{-k}D) = Op_1(a(x,\xi)\chi(h\xi)) = h^{-m}Op_h(\tilde{a}_h(x,\xi))$$

with

$$\tilde{a}_h(x,\xi) = h^m a(x,\xi/h)\chi(\xi) \quad \rightarrow \text{bounded family of } S^0.$$

- Short time heat kernel asymptotics: $e^{t\Delta_g}$ as $t \to 0$, $h = t^{1/2}$

ΨDO (part 1): Where can the *h* come from ?

- Semiclassical operators, e.g.

$$H(\hbar) = -\hbar^2\Delta + V(x) = Op_\hbar(|\xi|^2 + V(x))$$

- Large spectral parameter problem:

$$\begin{aligned} -\Delta + V(x) - \lambda e^{i\theta} &= \lambda(-\lambda^{-1}\Delta + \lambda^{-1}V(x) - e^{i\theta}) \\ &= \lambda\big(Op_h(|\xi|^2 + h^2V(x) - e^{i\theta})\big), \quad h = \frac{1}{\sqrt{\lambda}} \end{aligned}$$

- Regularity problems: if $\chi \in C_0^\infty(\mathbb{R}^n \setminus 0)$, $h = 2^{-k}$ is a *small* dyadic (Littlewood-Paley) parameter and $a \in S^m$,

$$a(x,D)\chi(2^{-k}D) = Op_1(a(x,\xi)\chi(h\xi)) = h^{-m}Op_h(\tilde{a}_h(x,\xi))$$

with

$$\tilde{a}_h(x,\xi) = h^m a(x,\xi/h)\chi(\xi) \quad \rightarrow \text{bounded family of } S^0.$$

- Short time heat kernel asymptotics: $e^{t\Delta_g}$ as $t \to 0$, $h = t^{1/2}$

ΨDO (part 1): Where can the *h* come from ?

- Semiclassical operators, e.g.

$$H(\hbar) = -\hbar^2\Delta + V(x) = Op_\hbar(|\xi|^2 + V(x))$$

- Large spectral parameter problem:

$$\begin{aligned} -\Delta + V(x) - \lambda e^{i\theta} &= \lambda(-\lambda^{-1}\Delta + \lambda^{-1}V(x) - e^{i\theta}) \\ &= \lambda\big(Op_h(|\xi|^2 + h^2 V(x) - e^{i\theta})\big), \quad h = \frac{1}{\sqrt{\lambda}} \end{aligned}$$

- Regularity problems: if $\chi \in C_0^\infty(\mathbb{R}^n \setminus 0)$, $h = 2^{-k}$ is a *small* dyadic (Littlewood-Paley) parameter and $a \in S^m$,

$$a(x, D)\chi(2^{-k}D) = Op_1(a(x, \xi)\chi(h\xi)) = h^{-m}Op_h(\tilde{a}_h(x, \xi))$$

with

$$\tilde{a}_h(x, \xi) = h^m a(x, \xi/h)\chi(\xi) \quad \rightarrow \text{bounded family of } S^0.$$

- Short time heat kernel asymptotics: $e^{t\Delta_g}$ as $t \to 0$, $h = t^{1/2}$

ΨDO (part 1): Where can the h come from ?

- Semiclassical operators, e.g.

$$H(\hbar) = -\hbar^2\Delta + V(x) = Op_\hbar(|\xi|^2 + V(x))$$

- Large spectral parameter problem:

$$\begin{aligned} -\Delta + V(x) - \lambda e^{i\theta} &= \lambda(-\lambda^{-1}\Delta + \lambda^{-1}V(x) - e^{i\theta}) \\ &= \lambda\big(Op_h(|\xi|^2 + h^2 V(x) - e^{i\theta})\big), \quad h = \frac{1}{\sqrt{\lambda}} \end{aligned}$$

- Regularity problems: if $\chi \in C_0^\infty(\mathbb{R}^n \setminus 0)$, $h = 2^{-k}$ is a *small* dyadic (Littlewood-Paley) parameter and $a \in S^m$,

$$a(x,D)\chi(2^{-k}D) = Op_1(a(x,\xi)\chi(h\xi)) = h^{-m}Op_h(\tilde{a}_h(x,\xi))$$

with

$$\tilde{a}_h(x,\xi) = h^m a(x,\xi/h)\chi(\xi) \quad \rightarrow \text{bounded family of } S^0.$$

- Short time heat kernel asymptotics: $e^{t\Delta_g}$ as $t \to 0$, $h = t^{1/2}$

ΨDO (part 1): Action on the Schwartz space

Proposition. For each h, the map $(a, u) \mapsto Op_h(a)u$ is continuous between $S^{\mu,m} \times \mathcal{S}$ and $\mathcal{S}$.

Proof.

$$|Op_h(a)u(x)| \lesssim_h \int \underbrace{\langle x\rangle^{\mu}\langle\xi\rangle^{m}}_{|a|\lesssim_h} \underbrace{\langle\xi\rangle^{-m-n-1}}_{|\hat{u}|\lesssim} d\xi \lesssim_h \langle x\rangle^{\mu}$$

and then more generally, for all $\alpha, \beta \in \mathbb{N}^n$

$$|x^{\alpha}\partial_x^{\beta} Op_h(a)u(x)| \lesssim \langle x\rangle^{\mu}$$

using (Exercise)

$$[\partial_j, Op_h(a)] = Op_h(\partial_{x_j} a), \qquad [x_j, Op_h(a)] = ihOp_h(\partial_{\xi_j} a).$$

ΨDO (part 1): Action on the Schwartz space

Proposition. For each h, the map $(a, u) \mapsto Op_h(a)u$ is continuous between $S^{\mu,m} \times \mathcal{S}$ and $\mathcal{S}$.

Proof.

$$|Op_h(a)u(x)| \lesssim_h \int \underbrace{\langle x\rangle^{\mu}\langle\xi\rangle^{m}}_{|a|\lesssim_h}\underbrace{\langle\xi\rangle^{-m-n-1}}_{|\hat{u}|\lesssim} d\xi \lesssim_h \langle x\rangle^{\mu}$$

and then more generally, for all $\alpha, \beta \in \mathbb{N}^n$

$$|x^{\alpha}\partial_x^{\beta} Op_h(a)u(x)| \lesssim \langle x\rangle^{\mu}$$

using (Exercise)

$$[\partial_j, Op_h(a)] = Op_h(\partial_{x_j} a), \qquad [x_j, Op_h(a)] = ihOp_h(\partial_{\xi_j} a).$$

ΨDO (part 1): Action on the Schwartz space

Proposition. For each h, the map $(a, u) \mapsto Op_h(a)u$ is continuous between $S^{\mu,m} \times \mathcal{S}$ and $\mathcal{S}$.

Proof.

$$|Op_h(a)u(x)| \lesssim_h \int \underbrace{\langle x\rangle^{\mu}\langle\xi\rangle^{m}}_{|a|\lesssim_h} \underbrace{\langle\xi\rangle^{-m-n-1}}_{|\hat{u}|\lesssim} d\xi \lesssim_h \langle x\rangle^{\mu}$$

and then more generally, for all $\alpha, \beta \in \mathbb{N}^n$

$$|x^{\alpha}\partial_x^{\beta} Op_h(a)u(x)| \lesssim \langle x\rangle^{\mu}$$

using (Exercise)

$$[\partial_j, Op_h(a)] = Op_h(\partial_{x_j} a), \qquad [x_j, Op_h(a)] = ihOp_h(\partial_{\xi_j} a).$$

ΨDO (part 1): Action on the Schwartz space

Proposition. For each h, the map $(a, u) \mapsto Op_h(a)u$ is continuous between $S^{\mu,m} \times \mathcal{S}$ and $\mathcal{S}$.

Proof.

$$|Op_h(a)u(x)| \lesssim_h \int \underbrace{\langle x\rangle^{\mu}\langle\xi\rangle^{m}}_{|a|\lesssim_h} \underbrace{\langle\xi\rangle^{-m-n-1}}_{|\hat{u}|\lesssim} d\xi \lesssim_h \langle x\rangle^{\mu}$$

and then more generally, for all $\alpha, \beta \in \mathbb{N}^n$

$$|x^{\alpha}\partial_x^{\beta} Op_h(a)u(x)| \lesssim \langle x\rangle^{\mu}$$

using (Exercise)

$$[\partial_j, Op_h(a)] = Op_h(\partial_{x_j} a), \qquad [x_j, Op_h(a)] = ihOp_h(\partial_{\xi_j} a).$$

ΨDO (part 1): Action on the Schwartz space

Proposition. For each h, the map $(a, u) \mapsto Op_h(a)u$ is continuous between $S^{\mu,m} \times \mathcal{S}$ and $\mathcal{S}$.

Proof.

$$|Op_h(a)u(x)| \lesssim_h \int \underbrace{\langle x\rangle^{\mu}\langle\xi\rangle^{m}}_{|a|\lesssim_h}\underbrace{\langle\xi\rangle^{-m-n-1}}_{|\hat{u}|\lesssim} d\xi \lesssim_h \langle x\rangle^{\mu}$$

and then more generally, for all $\alpha, \beta \in \mathbb{N}^n$

$$|x^{\alpha}\partial_x^{\beta} Op_h(a)u(x)| \lesssim \langle x\rangle^{\mu}$$

using (Exercise)

$$[\partial_j, Op_h(a)] = Op_h(\partial_{x_j} a), \qquad [x_j, Op_h(a)] = ihOp_h(\partial_{\xi_j} a).$$

ΨDO (part 1): Action on the Schwartz space

Proposition. For each h, the map $(a, u) \mapsto Op_h(a)u$ is continuous between $S^{\mu,m} \times \mathcal{S}$ and $\mathcal{S}$.

Proof.

$$|Op_h(a)u(x)| \lesssim_h \int \underbrace{\langle x\rangle^{\mu}\langle\xi\rangle^{m}}_{|a|\lesssim_h} \underbrace{\langle\xi\rangle^{-m-n-1}}_{|\hat{u}|\lesssim} d\xi \lesssim_h \langle x\rangle^{\mu}$$

and then more generally, for all $\alpha, \beta \in \mathbb{N}^n$

$$|x^{\alpha}\partial_x^{\beta} Op_h(a)u(x)| \lesssim \langle x\rangle^{\mu}$$

using (Exercise)

$$[\partial_j, Op_h(a)] = Op_h(\partial_{x_j} a), \qquad [x_j, Op_h(a)] = ihOp_h(\partial_{\xi_j} a).$$

ΨDO (part 1): Action on the Schwartz space

Proposition. For each h, the map $(a,u) \mapsto Op_h(a)u$ is continuous between $S^{\mu,m} \times \mathcal{S}$ and $\mathcal{S}$.
Proof.

$$|Op_h(a)u(x)| \lesssim_h \int \underbrace{\langle x\rangle^{\mu}\langle\xi\rangle^{m}}_{|a|\lesssim_h}\underbrace{\langle\xi\rangle^{-m-n-1}}_{|\hat{u}|\lesssim} d\xi \lesssim_h \langle x\rangle^{\mu}$$

and then more generally, for all $\alpha, \beta \in \mathbb{N}^n$

$$|x^{\alpha}\partial_x^{\beta} Op_h(a)u(x)| \lesssim \langle x\rangle^{\mu}$$

using (Exercise)

$$[\partial_j, Op_h(a)] = Op_h(\partial_{x_j} a), \qquad [x_j, Op_h(a)] = ihOp_h(\partial_{\xi_j} a).$$

ΨDO (part 1): Action on the Schwartz space

Proposition. For each h, the map $(a, u) \mapsto Op_h(a)u$ is continuous between $S^{\mu,m} \times \mathcal{S}$ and $\mathcal{S}$.

Proof.

$$|Op_h(a)u(x)| \lesssim_h \int \underbrace{\langle x\rangle^{\mu}\langle\xi\rangle^{m}}_{|a|\lesssim_h} \underbrace{\langle\xi\rangle^{-m-n-1}}_{|\hat{u}|\lesssim} d\xi \lesssim_h \langle x\rangle^{\mu}$$

and then more generally, for all $\alpha, \beta \in \mathbb{N}^n$

$$|x^{\alpha}\partial_x^{\beta} Op_h(a)u(x)| \lesssim \langle x\rangle^{\mu}$$

using (Exercise)

$$[\partial_j, Op_h(a)] = Op_h(\partial_{x_j} a), \qquad [x_j, Op_h(a)] = ihOp_h(\partial_{\xi_j} a).$$

ΨDO (part 1): Theorem of symbolic calculus

- **Composition.** For $a \in S^{\mu,m}$, $b \in S^{\mu',m'}$

$$Op_h(a)Op_h(b) = Op_h(c(h))$$

where $c(h) \sim \sum h^j c_j$ in $S^{\mu+\mu',m+m'}$, i.e. for all N

$$c(h) = \sum_{j<N} h^j c_j + h^N \tilde{c}_N(h), \qquad c_j := \sum_{|\alpha|=j} \frac{\partial_\xi^\alpha a D_x^\alpha b}{\alpha!} \in S^{\mu+\mu',m+m'-j}$$

and $\tilde{c}_N(h) \in S^{\mu+\mu',m+m'-N}$ depends continuously on (a,b).

- **Adjoint.** For $a \in S^{\mu,m}$,

$$(Op_h(a)v, u) = (v, Op_h(a^*(h))u), \qquad u, v \in \mathcal{S}$$

where

$$a^*(h) \sim \sum h^j a_j^* \text{ in } S^{\mu,m}, \qquad a_j^* = \sum_{|\alpha|=j} \frac{\partial_\xi^\alpha D_x^\alpha \overline{a}}{\alpha!} \in S^{\mu,m-j}.$$

ΨDO (part 1): Theorem of symbolic calculus

- **Composition.** For $a \in S^{\mu,m}$, $b \in S^{\mu',m'}$

$$Op_h(a)Op_h(b) = Op_h(c(h))$$

where $c(h) \sim \sum h^j c_j$ in $S^{\mu+\mu',m+m'}$, i.e. for all N

$$c(h) = \sum_{j<N} h^j c_j + h^N \tilde{c}_N(h), \qquad c_j := \sum_{|\alpha|=j} \frac{\partial_\xi^\alpha a D_x^\alpha b}{\alpha!} \in S^{\mu+\mu',m+m'-j}$$

and $\tilde{c}_N(h) \in S^{\mu+\mu',m+m'-N}$ depends continuously on (a,b).

- **Adjoint.** For $a \in S^{\mu,m}$,

$$(Op_h(a)v, u) = (v, Op_h(a^*(h))u), \qquad u, v \in \mathcal{S}$$

where

$$a^*(h) \sim \sum h^j a_j^* \text{ in } S^{\mu,m}, \qquad a_j^* = \sum_{|\alpha|=j} \frac{\partial_\xi^\alpha D_x^\alpha \overline{a}}{\alpha!} \in S^{\mu,m-j}.$$

ΨDO (part 1): Theorem of symbolic calculus

- **Composition.** For $a \in S^{\mu,m}$, $b \in S^{\mu',m'}$

$$Op_h(a)Op_h(b) = Op_h(c(h))$$

where $c(h) \sim \sum h^j c_j$ in $S^{\mu+\mu',m+m'}$, i.e. for all N

$$c(h) = \sum_{j<N} h^j c_j + h^N \tilde{c}_N(h), \qquad c_j := \sum_{|\alpha|=j} \frac{\partial_\xi^\alpha a D_x^\alpha b}{\alpha!} \in S^{\mu+\mu',m+m'-j}$$

and $\tilde{c}_N(h) \in S^{\mu+\mu',m+m'-N}$ depends continuously on (a, b).

- **Adjoint.** For $a \in S^{\mu,m}$,

$$(Op_h(a)v, u) = (v, Op_h(a^*(h))u), \qquad u, v \in \mathcal{S}$$

where

$$a^*(h) \sim \sum h^j a_j^* \text{ in } S^{\mu,m}, \qquad a_j^* = \sum_{|\alpha|=j} \frac{\partial_\xi^\alpha D_x^\alpha \overline{a}}{\alpha!} \in S^{\mu,m-j}.$$

ΨDO (part 1): Theorem of symbolic calculus

- **Composition.** For $a \in S^{\mu,m}$, $b \in S^{\mu',m'}$

$$Op_h(a)Op_h(b) = Op_h(c(h))$$

where $c(h) \sim \sum h^j c_j$ in $S^{\mu+\mu',m+m'}$, i.e. for all N

$$c(h) = \sum_{j<N} h^j c_j + h^N \tilde{c}_N(h), \qquad c_j := \sum_{|\alpha|=j} \frac{\partial_\xi^\alpha a D_x^\alpha b}{\alpha!} \in S^{\mu+\mu',m+m'-j}$$

and $\tilde{c}_N(h) \in S^{\mu+\mu',m+m'-N}$ depends continuously on (a, b).

- **Adjoint.** For $a \in S^{\mu,m}$,

$$(Op_h(a)v, u) = (v, Op_h(a^*(h))u), \qquad u, v \in \mathcal{S}$$

where

$$a^*(h) \sim \sum h^j a_j^* \text{ in } S^{\mu,m}, \qquad a_j^* = \sum_{|\alpha|=j} \frac{\partial_\xi^\alpha D_x^\alpha \overline{a}}{\alpha!} \in S^{\mu,m-j}.$$

ΨDO (part 1): Theorem of symbolic calculus

- **Composition.** For $a \in S^{\mu,m}$, $b \in S^{\mu',m'}$

$$Op_h(a)Op_h(b) = Op_h(c(h))$$

where $c(h) \sim \sum h^j c_j$ in $S^{\mu+\mu',m+m'}$, i.e. for all N

$$c(h) = \sum_{j<N} h^j c_j + h^N \tilde{c}_N(h), \qquad c_j := \sum_{|\alpha|=j} \frac{\partial_\xi^\alpha a D_x^\alpha b}{\alpha!} \in S^{\mu+\mu',m+m'-j}$$

and $\tilde{c}_N(h) \in S^{\mu+\mu',m+m'-N}$ depends continuously on (a,b).

- **Adjoint.** For $a \in S^{\mu,m}$,

$$(Op_h(a)v, u) = (v, Op_h(a^*(h))u), \qquad u, v \in \mathcal{S}$$

where

$$a^*(h) \sim \sum h^j a_j^* \text{ in } S^{\mu,m}, \qquad a_j^* = \sum_{|\alpha|=j} \frac{\partial_\xi^\alpha D_x^\alpha \overline{a}}{\alpha!} \in S^{\mu,m-j}.$$

ΨDO (part 1): Theorem of symbolic calculus

- **Composition.** For $a \in S^{\mu,m}$, $b \in S^{\mu',m'}$

$$Op_h(a)Op_h(b) = Op_h(c(h))$$

where $c(h) \sim \sum h^j c_j$ in $S^{\mu+\mu',m+m'}$, i.e. for all N

$$c(h) = \sum_{j<N} h^j c_j + h^N \tilde{c}_N(h), \qquad c_j := \sum_{|\alpha|=j} \frac{\partial_\xi^\alpha a D_x^\alpha b}{\alpha!} \in S^{\mu+\mu',m+m'-j}$$

and $\tilde{c}_N(h) \in S^{\mu+\mu',m+m'-N}$ depends continuously on (a,b).

- **Adjoint.** For $a \in S^{\mu,m}$,

$$(Op_h(a)v, u) = (v, Op_h(a^*(h))u), \qquad u, v \in \mathcal{S}$$

where

$$a^*(h) \sim \sum h^j a_j^* \text{ in } S^{\mu,m}, \qquad a_j^* = \sum_{|\alpha|=j} \frac{\partial_\xi^\alpha D_x^\alpha \overline{a}}{\alpha!} \in S^{\mu,m-j}.$$

ΨDO (part 1): Theorem of symbolic calculus

- **Composition.** For $a \in S^{\mu,m}$, $b \in S^{\mu',m'}$

$$Op_h(a)Op_h(b) = Op_h(c(h))$$

where $c(h) \sim \sum h^j c_j$ in $S^{\mu+\mu',m+m'}$, i.e. for all N

$$c(h) = \sum_{j<N} h^j c_j + h^N \tilde{c}_N(h), \qquad c_j := \sum_{|\alpha|=j} \frac{\partial_\xi^\alpha a D_x^\alpha b}{\alpha!} \in S^{\mu+\mu',m+m'-j}$$

and $\tilde{c}_N(h) \in S^{\mu+\mu',m+m'-N}$ depends continuously on (a, b).

- **Adjoint.** For $a \in S^{\mu,m}$,

$$(Op_h(a)v, u) = (v, Op_h(a^*(h))u), \qquad u, v \in \mathcal{S}$$

where

$$a^*(h) \sim \sum h^j a_j^* \text{ in } S^{\mu,m}, \qquad a_j^* = \sum_{|\alpha|=j} \frac{\partial_\xi^\alpha D_x^\alpha \bar{a}}{\alpha!} \in S^{\mu,m-j}.$$

ΨDO (part 1): Theorem of symbolic calculus

- **Composition.** For $a \in S^{\mu,m}$, $b \in S^{\mu',m'}$

$$Op_h(a)Op_h(b) = Op_h(c(h))$$

where $c(h) \sim \sum h^j c_j$ in $S^{\mu+\mu',m+m'}$, i.e. for all N

$$c(h) = \sum_{j<N} h^j c_j + h^N \tilde{c}_N(h), \qquad c_j := \sum_{|\alpha|=j} \frac{\partial_\xi^\alpha a D_x^\alpha b}{\alpha!} \in S^{\mu+\mu',m+m'-j}$$

and $\tilde{c}_N(h) \in S^{\mu+\mu',m+m'-N}$ depends continuously on (a, b).

- **Adjoint.** For $a \in S^{\mu,m}$,

$$\big(Op_h(a)v, u\big) = \big(v, Op_h(a^*(h))u\big), \qquad u, v \in \mathcal{S}$$

where

$$a^*(h) \sim \sum h^j a_j^* \text{ in } S^{\mu,m}, \qquad a_j^* = \sum_{|\alpha|=j} \frac{\partial_\xi^\alpha D_x^\alpha \overline{a}}{\alpha!} \in S^{\mu,m-j}.$$

ΨDO (part 1): Theorem of symbolic calculus

- **Composition.** For $a \in S^{\mu,m}$, $b \in S^{\mu',m'}$

$$Op_h(a)Op_h(b) = Op_h(c(h))$$

where $c(h) \sim \sum h^j c_j$ in $S^{\mu+\mu',m+m'}$, i.e. for all N

$$c(h) = \sum_{j<N} h^j c_j + h^N \tilde{c}_N(h), \qquad c_j := \sum_{|\alpha|=j} \frac{\partial_\xi^\alpha a D_x^\alpha b}{\alpha!} \in S^{\mu+\mu',m+m'-j}$$

and $\tilde{c}_N(h) \in S^{\mu+\mu',m+m'-N}$ depends continuously on (a,b).

- **Adjoint.** For $a \in S^{\mu,m}$,

$$\big(Op_h(a)v, u\big) = \big(v, Op_h(a^*(h))u\big), \qquad u, v \in \mathcal{S}$$

where

$$a^*(h) \sim \sum h^j a_j^* \text{ in } S^{\mu,m}, \qquad a_j^* = \sum_{|\alpha|=j} \frac{\partial_\xi^\alpha D_x^\alpha \overline{a}}{\alpha!} \in S^{\mu,m-j}.$$

ΨDO (part 1): L^2 boundedness

Corollary. For any $a \in S^{\mu,m}$, $Op_h(a)$ extends from $\mathcal{S}$ to $\mathcal{S}'$ by

$$\big(Op_h(a)T, u\big) := \big(T, Op_h(a^*(h))u\big), \qquad T \in \mathcal{S}',\ u \in \mathcal{S}.$$

Theorem (L^2 boundedness). If $a \in S^{0,0} (= S^0)$, $Op_h(a)$ is bounded on L^2. More precisely, there exist C and N such that

$$\big|\big|Op_h(a)u\big|\big|_{L^2} \leq C||a||_{S^{0,0},N}||u||_{L^2}$$

for all $h \in (0, 1]$, all $a \in S^{0,0}$ and all $u \in \mathcal{S}$ (or $u \in L^2$).

Corollary

- If $\mu < 0$, $m < 0$ and $a \in S^{\mu,m}$, $Op_h(a)$ is compact on L^2.
- If $\mu = 0$, $m \in \mathbb{R}$ and $a \in S^{0,m} = S^m$, then $Op_h(a)$ is bounded from H^s to H^{s-m} for every $s \in \mathbb{R}$.

ΨDO (part 1): L^2 boundedness

Corollary. For any $a \in S^{\mu,m}$, $Op_h(a)$ extends from $\mathcal{S}$ to $\mathcal{S}'$ by

$$\big(Op_h(a)T, u\big) := \big(T, Op_h(a^*(h))u\big), \qquad T \in \mathcal{S}',\ u \in \mathcal{S}.$$

Theorem (L^2 boundedness). If $a \in S^{0,0}(= S^0)$, $Op_h(a)$ is bounded on L^2. More precisely, there exist C and N such that

$$\big|\big|Op_h(a)u\big|\big|_{L^2} \leq C||a||_{S^{0,0},N}||u||_{L^2}$$

for all $h \in (0,1]$, all $a \in S^{0,0}$ and all $u \in \mathcal{S}$ (or $u \in L^2$).

Corollary

- If $\mu < 0$, $m < 0$ and $a \in S^{\mu,m}$, $Op_h(a)$ is compact on L^2.
- If $\mu = 0$, $m \in \mathbb{R}$ and $a \in S^{0,m} = S^m$, then $Op_h(a)$ is bounded from H^s to H^{s-m} for every $s \in \mathbb{R}$.

ΨDO (part 1): L^2 boundedness

Corollary. For any $a \in S^{\mu,m}$, $Op_h(a)$ extends from $\mathcal{S}$ to $\mathcal{S}'$ by

$$\big(Op_h(a)T, u\big) := \big(T, Op_h(a^*(h))u\big), \qquad T \in \mathcal{S}', \ u \in \mathcal{S}.$$

Theorem (L^2 boundedness). If $a \in S^{0,0} (= S^0)$, $Op_h(a)$ is bounded on L^2. More precisely, there exist C and N such that

$$\big|\big|Op_h(a)u\big|\big|_{L^2} \leq C||a||_{S^{0,0},N}||u||_{L^2}$$

for all $h \in (0, 1]$, all $a \in S^{0,0}$ and all $u \in \mathcal{S}$ (or $u \in L^2$).

Corollary

- If $\mu < 0$, $m < 0$ and $a \in S^{\mu,m}$, $Op_h(a)$ is compact on L^2.
- If $\mu = 0$, $m \in \mathbb{R}$ and $a \in S^{0,m} = S^m$, then $Op_h(a)$ is bounded from H^s to H^{s-m} for every $s \in \mathbb{R}$.

ΨDO (part 1): L^2 boundedness

Corollary. For any $a \in S^{\mu,m}$, $Op_h(a)$ extends from $\mathcal{S}$ to $\mathcal{S}'$ by

$$\big(Op_h(a)T, u\big) := \big(T, Op_h(a^*(h))u\big), \qquad T \in \mathcal{S}',\ u \in \mathcal{S}.$$

Theorem (L^2 boundedness). If $a \in S^{0,0} (= S^0)$, $Op_h(a)$ is bounded on L^2. More precisely, there exist C and N such that

$$\big|\big|Op_h(a)u\big|\big|_{L^2} \leq C||a||_{S^{0,0},N}||u||_{L^2}$$

for all $h \in (0,1]$, all $a \in S^{0,0}$ and all $u \in \mathcal{S}$ (or $u \in L^2$).

Corollary

- If $\mu < 0$, $m < 0$ and $a \in S^{\mu,m}$, $Op_h(a)$ is compact on L^2.
- If $\mu = 0$, $m \in \mathbb{R}$ and $a \in S^{0,m} = S^m$, then $Op_h(a)$ is bounded from H^s to H^{s-m} for every $s \in \mathbb{R}$.

ΨDO (part 1): L^2 boundedness

Corollary. For any $a \in S^{\mu,m}$, $Op_h(a)$ extends from $\mathcal{S}$ to $\mathcal{S}'$ by

$$\big(Op_h(a)T, u\big) := \big(T, Op_h(a^*(h))u\big), \qquad T \in \mathcal{S}', \ u \in \mathcal{S}.$$

Theorem (L^2 boundedness). If $a \in S^{0,0} (= S^0)$, $Op_h(a)$ is bounded on L^2. More precisely, there exist C and N such that

$$\big|\big|Op_h(a)u\big|\big|_{L^2} \leq C||a||_{S^{0,0},N}||u||_{L^2}$$

for all $h \in (0,1]$, all $a \in S^{0,0}$ and all $u \in \mathcal{S}$ (or $u \in L^2$).

Corollary

- If $\mu < 0$, $m < 0$ and $a \in S^{\mu,m}$, $Op_h(a)$ is compact on L^2.
- If $\mu = 0$, $m \in \mathbb{R}$ and $a \in S^{0,m} = S^m$, then $Op_h(a)$ is bounded from H^s to H^{s-m} for every $s \in \mathbb{R}$.

ΨDO (part 1): L^2 boundedness

Corollary. For any $a \in S^{\mu,m}$, $Op_h(a)$ extends from $\mathcal{S}$ to $\mathcal{S}'$ by

$$\big(Op_h(a)T, u\big) := \big(T, Op_h(a^*(h))u\big), \qquad T \in \mathcal{S}',\ u \in \mathcal{S}.$$

Theorem (L^2 boundedness). If $a \in S^{0,0} (= S^0)$, $Op_h(a)$ is bounded on L^2. More precisely, there exist C and N such that

$$\big|\big|Op_h(a)u\big|\big|_{L^2} \leq C||a||_{S^{0,0},N}||u||_{L^2}$$

for all $h \in (0,1]$, all $a \in S^{0,0}$ and all $u \in \mathcal{S}$ (or $u \in L^2$).

Corollary

- If $\mu < 0$, $m < 0$ and $a \in S^{\mu,m}$, $Op_h(a)$ is compact on L^2.
- If $\mu = 0$, $m \in \mathbb{R}$ and $a \in S^{0,m} = S^m$, then $Op_h(a)$ is bounded from H^s to H^{s-m} for every $s \in \mathbb{R}$.

ΨDO (part 1): L^2 boundedness

Corollary. For any $a \in S^{\mu,m}$, $Op_h(a)$ extends from $\mathcal{S}$ to $\mathcal{S}'$ by

$$\big(Op_h(a)T, u\big) := \big(T, Op_h(a^*(h))u\big), \qquad T \in \mathcal{S}', \ u \in \mathcal{S}.$$

Theorem (L^2 boundedness). If $a \in S^{0,0} (= S^0)$, $Op_h(a)$ is bounded on L^2. More precisely, there exist C and N such that

$$\big|\big|Op_h(a)u\big|\big|_{L^2} \leq C||a||_{S^{0,0},N}||u||_{L^2}$$

for all $h \in (0, 1]$, all $a \in S^{0,0}$ and all $u \in \mathcal{S}$ (or $u \in L^2$).

Corollary

- If $\mu < 0$, $m < 0$ and $a \in S^{\mu,m}$, $Op_h(a)$ is compact on L^2.
- If $\mu = 0$, $m \in \mathbb{R}$ and $a \in S^{0,m} = S^m$, then $Op_h(a)$ is bounded from H^s to H^{s-m} for every $s \in \mathbb{R}$.

ΨDO (part 1): L^2 boundedness

Corollary. For any $a \in S^{\mu,m}$, $Op_h(a)$ extends from $\mathcal{S}$ to $\mathcal{S}'$ by

$$\big(Op_h(a)T, u\big) := \big(T, Op_h(a^*(h))u\big), \qquad T \in \mathcal{S}',\ u \in \mathcal{S}.$$

Theorem (L^2 boundedness). If $a \in S^{0,0}(= S^0)$, $Op_h(a)$ is bounded on L^2. More precisely, there exist C and N such that

$$\big|\big|Op_h(a)u\big|\big|_{L^2} \leq C||a||_{S^{0,0},N}||u||_{L^2}$$

for all $h \in (0,1]$, all $a \in S^{0,0}$ and all $u \in \mathcal{S}$ (or $u \in L^2$).

Corollary

- If $\mu < 0$, $m < 0$ and $a \in S^{\mu,m}$, $Op_h(a)$ is compact on L^2.
- If $\mu = 0$, $m \in \mathbb{R}$ and $a \in S^{0,m} = S^m$, then $Op_h(a)$ is bounded from H^s to H^{s-m} for every $s \in \mathbb{R}$.

ΨDO (part 1): Proof of the Sobolev boundedness

$$\begin{aligned} ||a(x,D)u||_{H^{s-m}} &= ||\langle D\rangle^{s-m}a(x,D)u||_{L^2} \\ &= ||\langle D\rangle^{s-m}a(x,D)\langle D\rangle^{-s}\langle D\rangle^{s}u||_{L^2} \\ &= ||b(x,D)a(x,D)c(x,D)\langle D\rangle^{s}u||_{L^2} \end{aligned}$$

with $b(x,\xi)=\langle\xi\rangle^{s-m}\in S^{s-m}$ and $c(x,\xi)=\langle\xi\rangle^{-s}\in S^{-s}$ so

$$b(x,D)a(x,D)c(x,D)=d(x,D), \qquad d\in S^{s-m+m-s}=S^0.$$

Then

$$\begin{aligned} ||a(x,D)u||_{H^{s-m}} &\leq ||d(x,D)||_{L^2\to L^2}||\langle D\rangle^{s}u||_{L^2} \\ &\leq C||u||_{H^s}. \end{aligned}$$

ΨDO (part 1): Proof of the Sobolev boundedness

$$\begin{aligned} ||a(x,D)u||_{H^{s-m}} &= ||\langle D\rangle^{s-m} a(x,D)u||_{L^2} \\ &= ||\langle D\rangle^{s-m} a(x,D)\langle D\rangle^{-s}\langle D\rangle^{s} u||_{L^2} \\ &= ||b(x,D)a(x,D)c(x,D)\langle D\rangle^{s} u||_{L^2} \end{aligned}$$

with $b(x,\xi) = \langle\xi\rangle^{s-m} \in S^{s-m}$ and $c(x,\xi) = \langle\xi\rangle^{-s} \in S^{-s}$ so

$$b(x,D)a(x,D)c(x,D) = d(x,D), \qquad d \in S^{s-m+m-s} = S^0.$$

Then

$$\begin{aligned} ||a(x,D)u||_{H^{s-m}} &\leq ||d(x,D)||_{L^2\to L^2} ||\langle D\rangle^{s} u||_{L^2} \\ &\leq C||u||_{H^s}. \end{aligned}$$

ΨDO (part 1): Proof of the Sobolev boundedness

$$
\begin{aligned}
||a(x,D)u||_{H^{s-m}} &= ||\langle D\rangle^{s-m} a(x,D)u||_{L^2} \\
&= ||\langle D\rangle^{s-m} a(x,D)\langle D\rangle^{-s}\langle D\rangle^{s} u||_{L^2} \\
&= ||b(x,D)a(x,D)c(x,D)\langle D\rangle^{s} u||_{L^2}
\end{aligned}
$$

with $b(x,\xi) = \langle\xi\rangle^{s-m} \in S^{s-m}$ and $c(x,\xi) = \langle\xi\rangle^{-s} \in S^{-s}$ so

$$
b(x,D)a(x,D)c(x,D) = d(x,D), \qquad d \in S^{s-m+m-s} = S^0.
$$

Then

$$
\begin{aligned}
||a(x,D)u||_{H^{s-m}} &\leq ||d(x,D)||_{L^2\to L^2} ||\langle D\rangle^{s} u||_{L^2} \\
&\leq C||u||_{H^s}.
\end{aligned}
$$

ΨDO (part 1): Proof of the Sobolev boundedness

$$
\begin{aligned}
||a(x,D)u||_{H^{s-m}} &= ||\langle D\rangle^{s-m}a(x,D)u||_{L^2} \\
&= ||\langle D\rangle^{s-m}a(x,D)\langle D\rangle^{-s}\langle D\rangle^{s}u||_{L^2} \\
&= ||b(x,D)a(x,D)c(x,D)\langle D\rangle^{s}u||_{L^2}
\end{aligned}
$$

with $b(x,\xi) = \langle\xi\rangle^{s-m} \in S^{s-m}$ and $c(x,\xi) = \langle\xi\rangle^{-s} \in S^{-s}$ so

$$
b(x,D)a(x,D)c(x,D) = d(x,D), \qquad d \in S^{s-m+m-s} = S^0.
$$

Then

$$
\begin{aligned}
||a(x,D)u||_{H^{s-m}} &\le ||d(x,D)||_{L^2\to L^2}||\langle D\rangle^{s}u||_{L^2} \\
&\le C||u||_{H^s}.
\end{aligned}
$$

ΨDO (part 1): Proof of the Sobolev boundedness

$$
\begin{aligned}
||a(x,D)u||_{H^{s-m}} &= ||\langle D\rangle^{s-m}a(x,D)u||_{L^2} \\
&= ||\langle D\rangle^{s-m}a(x,D)\langle D\rangle^{-s}\langle D\rangle^{s}u||_{L^2} \\
&= ||b(x,D)a(x,D)c(x,D)\langle D\rangle^{s}u||_{L^2}
\end{aligned}
$$

with $b(x,\xi)=\langle\xi\rangle^{s-m}\in S^{s-m}$ and $c(x,\xi)=\langle\xi\rangle^{-s}\in S^{-s}$ so

$$
b(x,D)a(x,D)c(x,D)=d(x,D), \qquad d\in S^{s-m+m-s}=S^{0}.
$$

Then

$$
\begin{aligned}
||a(x,D)u||_{H^{s-m}} &\le ||d(x,D)||_{L^2\to L^2}||\langle D\rangle^{s}u||_{L^2} \\
&\le C||u||_{H^{s}}.
\end{aligned}
$$

ΨDO (part 1): Proof of the Sobolev boundedness

$$\begin{aligned}||a(x,D)u||_{H^{s-m}} &= ||\langle D\rangle^{s-m}a(x,D)u||_{L^2}\\ &= ||\langle D\rangle^{s-m}a(x,D)\langle D\rangle^{-s}\langle D\rangle^{s}u||_{L^2}\\ &= ||b(x,D)a(x,D)c(x,D)\langle D\rangle^{s}u||_{L^2}\end{aligned}$$

with $b(x,\xi)=\langle\xi\rangle^{s-m}\in S^{s-m}$ and $c(x,\xi)=\langle\xi\rangle^{-s}\in S^{-s}$ so

$$b(x,D)a(x,D)c(x,D)=d(x,D),\qquad d\in S^{s-m+m-s}=S^0.$$

Then

$$\begin{aligned}||a(x,D)u||_{H^{s-m}} &\le ||d(x,D)||_{L^2\to L^2}||\langle D\rangle^{s}u||_{L^2}\\ &\le C||u||_{H^s}.\end{aligned}$$

ΨDO (part 1): Proof of the Sobolev boundedness

$$\begin{aligned} ||a(x,D)u||_{H^{s-m}} &= ||\langle D\rangle^{s-m}a(x,D)u||_{L^2} \\ &= ||\langle D\rangle^{s-m}a(x,D)\langle D\rangle^{-s}\langle D\rangle^{s}u||_{L^2} \\ &= ||b(x,D)a(x,D)c(x,D)\langle D\rangle^{s}u||_{L^2} \end{aligned}$$

with $b(x,\xi) = \langle\xi\rangle^{s-m} \in S^{s-m}$ and $c(x,\xi) = \langle\xi\rangle^{-s} \in S^{-s}$ so

$$b(x,D)a(x,D)c(x,D) = d(x,D), \qquad d \in S^{s-m+m-s} = S^0.$$

Then

$$\begin{aligned} ||a(x,D)u||_{H^{s-m}} &\leq ||d(x,D)||_{L^2\to L^2}||\langle D\rangle^{s}u||_{L^2} \\ &\leq C||u||_{H^s}. \end{aligned}$$

ΨDO (part 1): Proof of the Sobolev boundedness

$$\begin{aligned} ||a(x,D)u||_{H^{s-m}} &= ||\langle D\rangle^{s-m} a(x,D)u||_{L^2} \\ &= ||\langle D\rangle^{s-m} a(x,D)\langle D\rangle^{-s}\langle D\rangle^{s} u||_{L^2} \\ &= ||b(x,D)a(x,D)c(x,D)\langle D\rangle^{s} u||_{L^2} \end{aligned}$$

with $b(x,\xi) = \langle\xi\rangle^{s-m} \in S^{s-m}$ and $c(x,\xi) = \langle\xi\rangle^{-s} \in S^{-s}$ so

$$b(x,D)a(x,D)c(x,D) = d(x,D), \qquad d \in S^{s-m+m-s} = S^0.$$

Then

$$\begin{aligned} ||a(x,D)u||_{H^{s-m}} &\le ||d(x,D)||_{L^2\to L^2} ||\langle D\rangle^{s} u||_{L^2} \\ &\le C||u||_{H^s}. \end{aligned}$$

ΨDO (part 1): Proof of the Sobolev boundedness

$$\begin{aligned}
||a(x,D)u||_{H^{s-m}} &= ||\langle D\rangle^{s-m}a(x,D)u||_{L^2} \\
&= ||\langle D\rangle^{s-m}a(x,D)\langle D\rangle^{-s}\langle D\rangle^{s}u||_{L^2} \\
&= ||b(x,D)a(x,D)c(x,D)\langle D\rangle^{s}u||_{L^2}
\end{aligned}$$

with $b(x,\xi) = \langle\xi\rangle^{s-m} \in S^{s-m}$ and $c(x,\xi) = \langle\xi\rangle^{-s} \in S^{-s}$ so

$$b(x,D)a(x,D)c(x,D) = d(x,D), \qquad d \in S^{s-m+m-s} = S^0.$$

Then

$$\begin{aligned}
||a(x,D)u||_{H^{s-m}} &\le ||d(x,D)||_{L^2\to L^2}||\langle D\rangle^{s}u||_{L^2} \\
&\le C||u||_{H^s}.
\end{aligned}$$

ΨDO (part 1): Proof of the Sobolev boundedness

$$\begin{aligned}||a(x,D)u||_{H^{s-m}} &= ||\langle D\rangle^{s-m}a(x,D)u||_{L^2}\\ &= ||\langle D\rangle^{s-m}a(x,D)\langle D\rangle^{-s}\langle D\rangle^{s}u||_{L^2}\\ &= ||b(x,D)a(x,D)c(x,D)\langle D\rangle^{s}u||_{L^2}\end{aligned}$$

with $b(x,\xi)=\langle\xi\rangle^{s-m}\in S^{s-m}$ and $c(x,\xi)=\langle\xi\rangle^{-s}\in S^{-s}$ so

$$b(x,D)a(x,D)c(x,D)=d(x,D), \qquad d\in S^{s-m+m-s}=S^0.$$

Then

$$\begin{aligned}||a(x,D)u||_{H^{s-m}} &\leq ||d(x,D)||_{L^2\to L^2}||\langle D\rangle^{s}u||_{L^2}\\ &\leq C||u||_{H^s}.\end{aligned}$$

ΨDO (part 1): Proof of the Sobolev boundedness

$$\begin{aligned}||a(x,D)u||_{H^{s-m}} &= ||\langle D\rangle^{s-m}a(x,D)u||_{L^2}\\ &= ||\langle D\rangle^{s-m}a(x,D)\langle D\rangle^{-s}\langle D\rangle^{s}u||_{L^2}\\ &= ||b(x,D)a(x,D)c(x,D)\langle D\rangle^{s}u||_{L^2}\end{aligned}$$

with $b(x,\xi)=\langle\xi\rangle^{s-m}\in S^{s-m}$ and $c(x,\xi)=\langle\xi\rangle^{-s}\in S^{-s}$ so

$$b(x,D)a(x,D)c(x,D)=d(x,D),\qquad d\in S^{s-m+m-s}=S^0.$$

Then

$$\begin{aligned}||a(x,D)u||_{H^{s-m}} &\leq ||d(x,D)||_{L^2\to L^2}||\langle D\rangle^{s}u||_{L^2}\\ &\leq C||u||_{H^s}.\end{aligned}$$

ΨDO (part 1): Proof of the Sobolev boundedness

$$\begin{aligned} ||a(x,D)u||_{H^{s-m}} &= ||\langle D\rangle^{s-m} a(x,D)u||_{L^2} \\ &= ||\langle D\rangle^{s-m} a(x,D)\langle D\rangle^{-s}\langle D\rangle^{s} u||_{L^2} \\ &= ||b(x,D)a(x,D)c(x,D)\langle D\rangle^{s} u||_{L^2} \end{aligned}$$

with $b(x,\xi) = \langle\xi\rangle^{s-m} \in S^{s-m}$ and $c(x,\xi) = \langle\xi\rangle^{-s} \in S^{-s}$ so

$$b(x,D)a(x,D)c(x,D) = d(x,D), \qquad d \in S^{s-m+m-s} = S^0.$$

Then

$$\begin{aligned} ||a(x,D)u||_{H^{s-m}} &\le ||d(x,D)||_{L^2\to L^2}||\langle D\rangle^{s} u||_{L^2} \\ &\le C||u||_{H^s}. \end{aligned}$$

ΨDO (part 1): Proof of the Sobolev boundedness

$$\begin{aligned} ||a(x,D)u||_{H^{s-m}} &= ||\langle D\rangle^{s-m} a(x,D)u||_{L^2} \\ &= ||\langle D\rangle^{s-m} a(x,D)\langle D\rangle^{-s}\langle D\rangle^{s} u||_{L^2} \\ &= ||b(x,D)a(x,D)c(x,D)\langle D\rangle^{s} u||_{L^2} \end{aligned}$$

with $b(x,\xi) = \langle\xi\rangle^{s-m} \in S^{s-m}$ and $c(x,\xi) = \langle\xi\rangle^{-s} \in S^{-s}$ so

$$b(x,D)a(x,D)c(x,D) = d(x,D), \qquad d \in S^{s-m+m-s} = S^0.$$

Then

$$\begin{aligned} ||a(x,D)u||_{H^{s-m}} &\le ||d(x,D)||_{L^2\to L^2}||\langle D\rangle^{s} u||_{L^2} \\ &\le C||u||_{H^s}. \end{aligned}$$

ΨDO (part 1): Proof of the Sobolev boundedness

$$\begin{aligned}
||a(x,D)u||_{H^{s-m}} &= ||\langle D\rangle^{s-m} a(x,D)u||_{L^2} \\
&= ||\langle D\rangle^{s-m} a(x,D)\langle D\rangle^{-s}\langle D\rangle^{s} u||_{L^2} \\
&= ||b(x,D)a(x,D)c(x,D)\langle D\rangle^{s} u||_{L^2}
\end{aligned}$$

with $b(x,\xi) = \langle\xi\rangle^{s-m} \in S^{s-m}$ and $c(x,\xi) = \langle\xi\rangle^{-s} \in S^{-s}$ so

$$b(x,D)a(x,D)c(x,D) = d(x,D), \qquad d \in S^{s-m+m-s} = S^0.$$

Then

$$\begin{aligned}
||a(x,D)u||_{H^{s-m}} &\leq ||d(x,D)||_{L^2\to L^2}||\langle D\rangle^{s} u||_{L^2} \\
&\leq C||u||_{H^s}.
\end{aligned}$$

ΨDO (part 1): Proof of the Sobolev boundedness

$$
\begin{aligned}
||a(x,D)u||_{H^{s-m}} &= ||\langle D\rangle^{s-m} a(x,D)u||_{L^2} \\
&= ||\langle D\rangle^{s-m} a(x,D)\langle D\rangle^{-s}\langle D\rangle^{s} u||_{L^2} \\
&= ||b(x,D)a(x,D)c(x,D)\langle D\rangle^{s} u||_{L^2}
\end{aligned}
$$

with $b(x,\xi) = \langle\xi\rangle^{s-m} \in S^{s-m}$ and $c(x,\xi) = \langle\xi\rangle^{-s} \in S^{-s}$ so

$$
b(x,D)a(x,D)c(x,D) = d(x,D), \qquad d \in S^{s-m+m-s} = S^{0}.
$$

Then

$$
\begin{aligned}
||a(x,D)u||_{H^{s-m}} &\leq ||d(x,D)||_{L^2\to L^2} ||\langle D\rangle^{s} u||_{L^2} \\
&\leq C||u||_{H^s}.
\end{aligned}
$$

ΨDO (part 1): Proof of the Sobolev boundedness

$$\begin{aligned}||a(x,D)u||_{H^{s-m}} &= ||\langle D\rangle^{s-m}a(x,D)u||_{L^2}\\ &= ||\langle D\rangle^{s-m}a(x,D)\langle D\rangle^{-s}\langle D\rangle^{s}u||_{L^2}\\ &= ||b(x,D)a(x,D)c(x,D)\langle D\rangle^{s}u||_{L^2}\end{aligned}$$

with $b(x,\xi)=\langle\xi\rangle^{s-m}\in S^{s-m}$ and $c(x,\xi)=\langle\xi\rangle^{-s}\in S^{-s}$ so

$$b(x,D)a(x,D)c(x,D)=d(x,D), \qquad d\in S^{s-m+m-s}=S^0.$$

Then

$$\begin{aligned}||a(x,D)u||_{H^{s-m}} &\le ||d(x,D)||_{L^2\to L^2}||\langle D\rangle^{s}u||_{L^2}\\ &\le C||u||_{H^s}.\end{aligned}$$

Interlude: Some references

Non-semiclassical
S. Alinhac-P. Gérard, *Op. pseudo. diff. et th. de Nash Moser*
G.B. Folland, *Harmonic analysis in phase space*
A. Grigis-J. Sjöstrand, *Microlocal analysis for differential op.*
L. Hörmander, *The Analysis of linear Partial Diff. Op. III*
N. Lerner, *Metrics on the Phase Space and Non-Selfadjoint Pseudo-Differential Operators*
M. Taylor, *Partial Differential Equations, II*

Semiclassical
M. Dimassi-J. Sjöstrand, *Spectral asymptotics in the semi-class. limit*
A. Martinez, *An intro. to semiclassical and microlocal analysis*
D. Robert, *Autour de l'approximation semi-classique*
M. Zworski, *Semiclassical analysis*

Application: Hörmander's Theorem on hypoelliptic operators

Consider a differential operator on $\mathbb{R}^n$ of the form

$$P = -\sum_{j=1}^{r} L_j^2 + L_0 + V,$$

with $L_0, L_1, \ldots, L_r$ real vector fields and V a smooth function.
Assumptions There exists N such that for all x,

$$\{L_k, [L_{k_1}, L_{k_2}], \ldots, [L_{k_1}, [L_{k_2}, \ldots [L_{k_{N-1}}, L_{k_N}]\}$$

span $T_x\mathbb{R}^n$. Assume also V and the L_k's coefficients are C_b^∞.
Example (local):

$$P = -\frac{\partial^2}{\partial x_1^2} - \sum_{j=2}^{n} x_1^{m_j} \frac{\partial}{\partial x_j}, \qquad 0 \le m_2 < \cdots < m_n.$$

Application: Hörmander's Theorem on hypoelliptic operators

Consider a differential operator on $\mathbb{R}^n$ of the form

$$P = -\sum_{j=1}^{r} L_j^2 + L_0 + V,$$

with $L_0, L_1, \ldots, L_r$ real vector fields and V a smooth function.

Assumptions There exists N such that for all x,

$$\{L_k, [L_{k_1}, L_{k_2}], \ldots, [L_{k_1}, [L_{k_2}, \ldots [L_{k_{N-1}}, L_{k_N}]\}$$

span $T_x\mathbb{R}^n$. Assume also V and the L_k's coefficients are C_b^∞.

Example (local):

$$P = -\frac{\partial^2}{\partial x_1^2} - \sum_{j=2}^{n} x_1^{m_j} \frac{\partial}{\partial x_j}, \qquad 0 \leq m_2 < \cdots < m_n.$$

Application: Hörmander's Theorem on hypoelliptic operators

Consider a differential operator on $\mathbb{R}^n$ of the form

$$P = -\sum_{j=1}^{r} L_j^2 + L_0 + V,$$

with $L_0, L_1, \ldots, L_r$ real vector fields and V a smooth function.
Assumptions There exists N such that for all x,

$$\{L_k, [L_{k_1}, L_{k_2}], \ldots, [L_{k_1}, [L_{k_2}, \ldots [L_{k_{N-1}}, L_{k_N}]]\}$$

span $T_x\mathbb{R}^n$. Assume also V and the L_k's coefficients are C_b^∞.
Example (local):

$$P = -\frac{\partial^2}{\partial x_1^2} - \sum_{j=2}^{n} x_1^{m_j} \frac{\partial}{\partial x_j}, \qquad 0 \leq m_2 < \cdots < m_n.$$

Application: Hörmander's Theorem on hypoelliptic operators

Consider a differential operator on $\mathbb{R}^n$ of the form

$$P = -\sum_{j=1}^{r} L_j^2 + L_0 + V,$$

with $L_0, L_1, \ldots, L_r$ real vector fields and V a smooth function.
Assumptions There exists N such that for all x,

$$\{L_k, [L_{k_1}, L_{k_2}], \ldots, [L_{k_1}, [L_{k_2}, \ldots [L_{k_{N-1}}, L_{k_N}]\}$$

span $T_x\mathbb{R}^n$. Assume also V and the L_k's coefficients are C_b^∞.

Example (local):

$$P = -\frac{\partial^2}{\partial x_1^2} - \sum_{j=2}^{n} x_1^{m_j} \frac{\partial}{\partial x_j}, \qquad 0 \leq m_2 < \cdots < m_n.$$

Application: Hörmander's Theorem on hypoelliptic operators

Consider a differential operator on $\mathbb{R}^n$ of the form

$$P = -\sum_{j=1}^{r} L_j^2 + L_0 + V,$$

with $L_0, L_1, \ldots, L_r$ real vector fields and V a smooth function.
Assumptions There exists N such that for all x,

$$\{L_k, [L_{k_1}, L_{k_2}], \ldots, [L_{k_1}, [L_{k_2}, \ldots [L_{k_{N-1}}, L_{k_N}]\}$$

span $T_x\mathbb{R}^n$. Assume also V and the L_k's coefficients are C_b^∞.
Example (local):

$$P = -\frac{\partial^2}{\partial x_1^2} - \sum_{j=2}^{n} x_1^{m_j} \frac{\partial}{\partial x_j}, \qquad 0 \leq m_2 < \cdots < m_n.$$

Application: Hörmander's Theorem on hypoelliptic operators

Here's an easy form of Hörmander's Theorem (we denote $||v||_{(s)} = ||v||_{H^s} = ||\langle D\rangle^s v||$):

Theorem. *There exists* $\epsilon \in (0, 1]$ *(*$\epsilon = 2^{-N}$*) such that for each* $\chi \in C_0^\infty(\mathbb{R}^n)$

$$||\chi u||_{(\epsilon)} \lesssim_\chi ||Pu||_{L^2} + ||u||_{L^2}$$

for all $u \in \mathcal{S}(\mathbb{R}^n)$.

Meaning. Even if P is not elliptic, having Pu (and u) in L^2 still gives a control on some (fractional) derivatives of u.

Interests.

- Can be used to prove the hypoellipticity of P: $Pu \in C^\infty \Rightarrow u \in C^\infty$
- On a compact manifold, it can be used to show that the resolvent of P is compact $\Rightarrow$ discrete spectrum.

Application: Hörmander's Theorem on hypoelliptic operators

Here's an easy form of Hörmander's Theorem (we denote $||v||_{(s)} = ||v||_{H^s} = ||\langle D \rangle^s v||$):

Theorem. *There exists* $\epsilon \in (0, 1]$ *(*$\epsilon = 2^{-N}$*) such that for each* $\chi \in C_0^\infty(\mathbb{R}^n)$

$$||\chi u||_{(\epsilon)} \lesssim_\chi ||Pu||_{L^2} + ||u||_{L^2}$$

for all $u \in \mathcal{S}(\mathbb{R}^n)$.

Meaning. Even if P is not elliptic, having Pu (and u) in L^2 still gives a control on some (fractional) derivatives of u.

Interests.

- Can be used to prove the hypoellipticity of P: $Pu \in C^\infty \Rightarrow u \in C^\infty$
- On a compact manifold, it can be used to show that the resolvent of P is compact $\Rightarrow$ discrete spectrum.

Application: Hörmander's Theorem on hypoelliptic operators

Here's an easy form of Hörmander's Theorem (we denote $||v||_{(s)} = ||v||_{H^s} = ||\langle D\rangle^s v||$):

Theorem. *There exists* $\epsilon \in (0, 1]$ *(*$\epsilon = 2^{-N}$*) such that for each* $\chi \in C_0^\infty(\mathbb{R}^n)$

$$||\chi u||_{(\epsilon)} \lesssim_\chi ||Pu||_{L^2} + ||u||_{L^2}$$

for all $u \in \mathcal{S}(\mathbb{R}^n)$.

Meaning. Even if P is not elliptic, having Pu (and u) in L^2 still gives a control on some (fractional) derivatives of u.

Interests.

- Can be used to prove the hypoellipticity of P: $Pu \in C^\infty \Rightarrow u \in C^\infty$
- On a compact manifold, it can be used to show that the resolvent of P is compact $\Rightarrow$ discrete spectrum.

Application: Hörmander's Theorem on hypoelliptic operators

Here's an easy form of Hörmander's Theorem (we denote $||v||_{(s)} = ||v||_{H^s} = ||\langle D\rangle^s v||$):

Theorem. *There exists* $\epsilon \in (0, 1]$ *(*$\epsilon = 2^{-N}$*) such that for each* $\chi \in C_0^\infty(\mathbb{R}^n)$

$$||\chi u||_{(\epsilon)} \lesssim_\chi ||Pu||_{L^2} + ||u||_{L^2}$$

for all $u \in \mathcal{S}(\mathbb{R}^n)$.

Meaning. Even if P is not elliptic, having Pu (and u) in L^2 still gives a control on some (fractional) derivatives of u.

Interests.

- Can be used to prove the hypoellipticity of P: $Pu \in C^\infty \Rightarrow u \in C^\infty$
- On a compact manifold, it can be used to show that the resolvent of P is compact $\Rightarrow$ discrete spectrum.

Application: Hörmander's Theorem on hypoelliptic operators

Here's an easy form of Hörmander's Theorem (we denote $||v||_{(s)} = ||v||_{H^s} = ||\langle D\rangle^s v||$):

Theorem. *There exists* $\epsilon \in (0, 1]$ *(*$\epsilon = 2^{-N}$*) such that for each* $\chi \in C_0^\infty(\mathbb{R}^n)$

$$||\chi u||_{(\epsilon)} \lesssim_\chi ||Pu||_{L^2} + ||u||_{L^2}$$

for all $u \in \mathcal{S}(\mathbb{R}^n)$.

Meaning. Even if P is not elliptic, having Pu (and u) in L^2 still gives a control on some (fractional) derivatives of u.

Interests.

- Can be used to prove the hypoellipticity of P: $Pu \in C^\infty \Rightarrow u \in C^\infty$
- On a compact manifold, it can be used to show that the resolvent of P is compact $\Rightarrow$ discrete spectrum.

Proof: First step

Rewrite

$$P = \sum_{j=1}^{r} L_j^* L_j + \tilde{L}_0 + V$$

with $\tilde{L}_0 = L_0 + \sum c_j L_j$ if $L_j^* = -L_j - c_j$. Note that

$$\tilde{L}_0^* = -\tilde{L}_0 + c_0, \qquad c_0 \in C_b^\infty(\mathbb{R}^n).$$

Now

$$\mathrm{Re}(Pu, u) = \sum_{j=1}^{r} ||L_j u||^2 + \mathrm{Re}(Vu, u) + \frac{((\tilde{L}_0 + \tilde{L}_0^*)u, u)}{2}$$

Thus

$$\sum_{j=1}^{r} ||L_j u||^2 \leq (Pu, u) + C||u||^2 \lesssim ||Pu||^2 + ||u||^2.$$

Proof: First step

Rewrite

$$P = \sum_{j=1}^{r} L_j^* L_j + \tilde{L}_0 + V$$

with $\tilde{L}_0 = L_0 + \sum c_j L_j$ if $L_j^* = -L_j - c_j$. Note that

$$\tilde{L}_0^* = -\tilde{L}_0 + c_0, \qquad c_0 \in C_b^\infty(\mathbb{R}^n).$$

Now

$$\mathrm{Re}(Pu,u) = \sum_{j=1}^{r} ||L_j u||^2 + \mathrm{Re}(Vu,u) + \frac{((\tilde{L}_0 + \tilde{L}_0^*)u,u)}{2}$$

Thus

$$\sum_{j=1}^{r} ||L_j u||^2 \leq (Pu,u) + C||u||^2 \lesssim ||Pu||^2 + ||u||^2.$$

Proof: First step

Rewrite

$$P = \sum_{j=1}^{r} L_j^* L_j + \tilde{L}_0 + V$$

with $\tilde{L}_0 = L_0 + \sum c_j L_j$ if $L_j^* = -L_j - c_j$. Note that

$$\tilde{L}_0^* = -\tilde{L}_0 + c_0, \qquad c_0 \in C_b^\infty(\mathbb{R}^n).$$

Now

$$\mathrm{Re}(Pu, u) = \sum_{j=1}^{r} ||L_j u||^2 + \mathrm{Re}(Vu, u) + \frac{((\tilde{L}_0 + \tilde{L}_0^*)u, u)}{2}$$

Thus

$$\sum_{j=1}^{r} ||L_j u||^2 \le (Pu, u) + C||u||^2 \lesssim ||Pu||^2 + ||u||^2.$$

Proof: First step

Rewrite

$$P = \sum_{j=1}^{r} L_j^* L_j + \tilde{L}_0 + V$$

with $\tilde{L}_0 = L_0 + \sum c_j L_j$ if $L_j^* = -L_j - c_j$. Note that

$$\tilde{L}_0^* = -\tilde{L}_0 + c_0, \qquad c_0 \in C_b^\infty(\mathbb{R}^n).$$

Now

$$\mathrm{Re}(Pu, u) = \sum_{j=1}^{r} ||L_j u||^2 + \mathrm{Re}(Vu, u) + \frac{((\tilde{L}_0 + \tilde{L}_0^*)u, u)}{2}$$

Thus

$$\sum_{j=1}^{r} ||L_j u||^2 \leq (Pu, u) + C||u||^2 \lesssim ||Pu||^2 + ||u||^2.$$

Proof: First step

Rewrite

$$P = \sum_{j=1}^{r} L_j^* L_j + \tilde{L}_0 + V$$

with $\tilde{L}_0 = L_0 + \sum c_j L_j$ if $L_j^* = -L_j - c_j$. Note that

$$\tilde{L}_0^* = -\tilde{L}_0 + c_0, \qquad c_0 \in C_b^\infty(\mathbb{R}^n).$$

Now

$$\mathrm{Re}(Pu, u) = \sum_{j=1}^{r} ||L_j u||^2 + \mathrm{Re}(Vu, u) + \frac{((\tilde{L}_0 + \tilde{L}_0^*)u, u)}{2}$$

Thus

$$\sum_{j=1}^{r} ||L_j u||^2 \leq (Pu, u) + C||u||^2 \lesssim ||Pu||^2 + ||u||^2.$$

Proof: Second step

We have seen $||L_j u||_{(0)} \lesssim ||Pu|| + ||u||$ $(1 \leq j \leq r)$. Now we show

$$||\tilde{L}_0 u||_{(-1/2)} \lesssim ||Pu|| + ||u||.$$

Note that

$$||L_j u||_{(-1/2)} \lesssim ||Pu|| + ||u||, \qquad 1 \leq j \leq r.$$

Let us decompose $P = P' + iP''$ with

$$P' = \frac{P + P^*}{2} = \sum_1^r L_j^* L_j + \mathrm{Re}(V)$$

$$P'' = \frac{P - P^*}{2i} = -i\tilde{L}_0 + \frac{\mathrm{div}(\tilde{L}_0)}{2i} + \mathrm{Im}(V)$$

Up to adding $C \gg 1$ to P we may assume that $P' \geq 0$ (see later). It also suffices to show

$$||P'' u||_{(-1/2)} \lesssim ||Pu|| + ||u||.$$

Proof: Second step

We have seen $||L_j u||_{(0)} \lesssim ||Pu|| + ||u||$ $(1 \le j \le r)$. Now we show

$$||\tilde{L}_0 u||_{(-1/2)} \lesssim ||Pu|| + ||u||.$$

Note that

$$||L_j u||_{(-1/2)} \lesssim ||Pu|| + ||u||, \qquad 1 \le j \le r.$$

Let us decompose $P = P' + iP''$ with

$$P' = \frac{P + P^*}{2} \quad = \quad \sum_1^r L_j^* L_j + \mathrm{Re}(V)$$

$$P'' = \frac{P - P^*}{2i} \quad = \quad -i\tilde{L}_0 + \frac{\mathrm{div}(\tilde{L}_0)}{2i} + \mathrm{Im}(V)$$

Up to adding $C \gg 1$ to P we may assume that $P' \ge 0$ (see later). It also suffices to show

$$||P'' u||_{(-1/2)} \lesssim ||Pu|| + ||u||.$$

Proof: Second step

We have seen $||L_j u||_{(0)} \lesssim ||Pu|| + ||u||$ $(1 \leq j \leq r)$. Now we show

$$||\tilde{L}_0 u||_{(-1/2)} \lesssim ||Pu|| + ||u||.$$

Note that

$$||L_j u||_{(-1/2)} \lesssim ||Pu|| + ||u||, \qquad 1 \leq j \leq r.$$

Let us decompose $P = P' + iP''$ with

$$P' = \frac{P + P^*}{2} = \sum_1^r L_j^* L_j + \mathrm{Re}(V)$$

$$P'' = \frac{P - P^*}{2i} = -i\tilde{L}_0 + \frac{\mathrm{div}(\tilde{L}_0)}{2i} + \mathrm{Im}(V)$$

Up to adding $C \gg 1$ to P we may assume that $P' \geq 0$ (see later). It also suffices to show

$$||P'' u||_{(-1/2)} \lesssim ||Pu|| + ||u||.$$

Proof: Second step

We have seen $||L_j u||_{(0)} \lesssim ||Pu|| + ||u||$ ($1 \leq j \leq r$). Now we show

$$||\tilde{L}_0 u||_{(-1/2)} \lesssim ||Pu|| + ||u||.$$

Note that

$$||L_j u||_{(-1/2)} \lesssim ||Pu|| + ||u||, \qquad 1 \leq j \leq r.$$

Let us decompose $P = P' + iP''$ with

$$P' = \frac{P + P^*}{2} \quad = \quad \sum_1^r L_j^* L_j + \mathrm{Re}(V)$$

$$P'' = \frac{P - P^*}{2i} \quad = \quad -i\tilde{L}_0 + \frac{\mathrm{div}(\tilde{L}_0)}{2i} + \mathrm{Im}(V)$$

Up to adding $C \gg 1$ to P we may assume that $P' \geq 0$ (see later). It also suffices to show

$$||P'' u||_{(-1/2)} \lesssim ||Pu|| + ||u||.$$

Proof: Second step

We have seen $||L_j u||_{(0)} \lesssim ||Pu|| + ||u||$ $(1 \leq j \leq r)$. Now we show

$$||\tilde{L}_0 u||_{(-1/2)} \lesssim ||Pu|| + ||u||.$$

Note that

$$||L_j u||_{(-1/2)} \lesssim ||Pu|| + ||u||, \qquad 1 \leq j \leq r.$$

Let us decompose $P = P' + iP''$ with

$$P' = \frac{P + P^*}{2} = \sum_1^r L_j^* L_j + \mathrm{Re}(V)$$

$$P'' = \frac{P - P^*}{2i} = -i\tilde{L}_0 + \frac{\mathrm{div}(\tilde{L}_0)}{2i} + \mathrm{Im}(V)$$

Up to adding $C \gg 1$ to P we may assume that $P' \geq 0$ (see later). It also suffices to show

$$||P'' u||_{(-1/2)} \lesssim ||Pu|| + ||u||.$$

Proof: Second step

We have seen $||L_j u||_{(0)} \lesssim ||Pu|| + ||u||$ $(1 \leq j \leq r)$. Now we show

$$||\tilde{L}_0 u||_{(-1/2)} \lesssim ||Pu|| + ||u||.$$

Note that

$$||L_j u||_{(-1/2)} \lesssim ||Pu|| + ||u||, \qquad 1 \leq j \leq r.$$

Let us decompose $P = P' + iP''$ with

$$P' = \frac{P + P^*}{2} \quad = \quad \sum_1^r L_j^* L_j + \mathrm{Re}(V)$$

$$P'' = \frac{P - P^*}{2i} \quad = \quad -i\tilde{L}_0 + \frac{\mathrm{div}(\tilde{L}_0)}{2i} + \mathrm{Im}(V)$$

Up to adding $C \gg 1$ to P we may assume that $P' \geq 0$ (see later). It also suffices to show

$$||P''u||_{(-1/2)} \lesssim ||Pu|| + ||u||.$$

Proof: Second step

We have seen $||L_j u||_{(0)} \lesssim ||Pu|| + ||u||$ $(1 \leq j \leq r)$. Now we show

$$||\tilde{L}_0 u||_{(-1/2)} \lesssim ||Pu|| + ||u||.$$

Note that

$$||L_j u||_{(-1/2)} \lesssim ||Pu|| + ||u||, \qquad 1 \leq j \leq r.$$

Let us decompose $P = P' + iP''$ with

$$P' = \frac{P + P^*}{2} = \sum_1^r L_j^* L_j + \mathrm{Re}(V)$$

$$P'' = \frac{P - P^*}{2i} = -i\tilde{L}_0 + \frac{\mathrm{div}(\tilde{L}_0)}{2i} + \mathrm{Im}(V)$$

Up to adding $C \gg 1$ to P we may assume that $P' \geq 0$ (see later). It also suffices to show

$$||P'' u||_{(-1/2)} \lesssim ||Pu|| + ||u||.$$

$$||P''u||^2_{(-\frac{1}{2})} \quad = \quad ||\langle D\rangle^{-1/2}P''u||^2 = (\underbrace{P''}_{=(P-P')/i} u, \underbrace{\langle D\rangle^{-1}P''}_{=:A\in Op(S^0)} u)$$

$$\leq \quad ||Pu||\,\underbrace{||Au||}_{\lesssim ||u||} + \left|(P'u, Au)\right|$$

Then, by Cauchy-Schwarz for the sesquilinear form $(P'v, w)$,

$$\left|(P'u, Au)\right| \leq \underbrace{(P'u, u)^{\frac{1}{2}}}_{(\mathrm{Re}(Pu,u))^{\frac{1}{2}}}(P'Au, Au)^{\frac{1}{2}}$$

while $2(P'Au, Au)$ reads $(PAu, Au) + (Au, PAu)$ i.e.

$$(APu, Au) + (Au, APu) + ([P, A]u, Au) + (Au, [P, A]u)$$

where

$$|(APu, Au)| \lesssim ||Pu||\ ||u||$$

$$||P''u||^2_{(-\frac{1}{2})} = ||\langle D\rangle^{-1/2}P''u||^2 = (\underbrace{P''}_{=(P-P')/i} u, \underbrace{\langle D\rangle^{-1}P''}_{=:A\in Op(S^0)} u)$$

$$\leq ||Pu||\,\underbrace{||Au||}_{\lesssim ||u||} + \left|(P'u, Au)\right|$$

Then, by Cauchy-Schwarz for the sesquilinear form $(P'v, w)$,

$$\left|(P'u, Au)\right| \leq \underbrace{(P'u, u)^{\frac{1}{2}}}_{(\mathrm{Re}(Pu,u))^{\frac{1}{2}}}(P'Au, Au)^{\frac{1}{2}}$$

while $2(P'Au, Au)$ reads $(PAu, Au) + (Au, PAu)$ i.e.

$$(APu, Au) + (Au, APu) + ([P, A]u, Au) + (Au, [P, A]u)$$

where

$$|(APu, Au)| \lesssim ||Pu||\;||u||$$

$$||P''u||^2_{(-\frac{1}{2})} = ||\langle D\rangle^{-1/2}P''u||^2 = (\underbrace{P''}_{=(P-P')/i} u, \underbrace{\langle D\rangle^{-1}P''}_{=:A\in Op(S^0)} u)$$

$$\leq ||Pu||\underbrace{||Au||}_{\lesssim||u||} + \left|(P'u, Au)\right|$$

Then, by Cauchy-Schwarz for the sesquilinear form $(P'v, w)$,

$$\left|(P'u, Au)\right| \leq \underbrace{(P'u, u)^{\frac{1}{2}}}_{(\mathrm{Re}(Pu,u))^{\frac{1}{2}}}(P'Au, Au)^{\frac{1}{2}}$$

while $2(P'Au, Au)$ reads $(PAu, Au) + (Au, PAu)$ i.e.

$$(APu, Au) + (Au, APu) + ([P, A]u, Au) + (Au, [P, A]u)$$

where

$$|(APu, Au)| \lesssim ||Pu||\ ||u||$$

$$||P''u||^2_{(-\frac{1}{2})} = ||\langle D\rangle^{-1/2}P''u||^2 = (\underbrace{P''}_{=(P-P')/i} u, \underbrace{\langle D\rangle^{-1}P''}_{=:A\in Op(S^0)} u)$$

$$\leq ||Pu|| \underbrace{||Au||}_{\lesssim ||u||} + |(P'u, Au)|$$

Then, by Cauchy-Schwarz for the sesquilinear form $(P'v, w)$,

$$|(P'u, Au)| \leq \underbrace{(P'u, u)^{\frac{1}{2}}}_{(\mathrm{Re}(Pu,u))^{\frac{1}{2}}} (P'Au, Au)^{\frac{1}{2}}$$

while $2(P'Au, Au)$ reads $(PAu, Au) + (Au, PAu)$ i.e.

$$(APu, Au) + (Au, APu) + ([P, A]u, Au) + (Au, [P, A]u)$$

where

$$|(APu, Au)| \lesssim ||Pu||\ ||u||$$

$$\|P''u\|^2_{(-\frac{1}{2})} = \|\langle D\rangle^{-1/2}P''u\|^2 = (\underbrace{P''}_{=(P-P')/i} u, \underbrace{\langle D\rangle^{-1}P''}_{=:A\in Op(S^0)} u)$$

$$\leq \|Pu\| \underbrace{\|Au\|}_{\lesssim\|u\|} + \left|(P'u, Au)\right|$$

Then, by Cauchy-Schwarz for the sesquilinear form $(P'v, w)$,

$$\left|(P'u, Au)\right| \leq \underbrace{(P'u, u)^{\frac{1}{2}}}_{(\mathrm{Re}(Pu,u))^{\frac{1}{2}}} (P'Au, Au)^{\frac{1}{2}}$$

while $2(P'Au, Au)$ reads $(PAu, Au) + (Au, PAu)$ i.e.

$$(APu, Au) + (Au, APu) + ([P, A]u, Au) + (Au, [P, A]u)$$

where

$$|(APu, Au)| \lesssim \|Pu\| \, \|u\|$$

$$
\begin{aligned}
||P''u||^2_{(-\frac{1}{2})} &= ||\langle D\rangle^{-1/2}P''u||^2 = (\underbrace{P''}_{=(P-P')/i} u, \underbrace{\langle D\rangle^{-1}P''}_{=:A\in Op(S^0)} u) \\
&\le ||Pu|| \underbrace{||Au||}_{\lesssim ||u||} + \left|(P'u, Au)\right|
\end{aligned}
$$

Then, by Cauchy-Schwarz for the sesquilinear form $(P'v, w)$,

$$
\left|(P'u, Au)\right| \le \underbrace{(P'u, u)^{\frac{1}{2}}}_{(\mathrm{Re}(Pu,u))^{\frac{1}{2}}} (P'Au, Au)^{\frac{1}{2}}
$$

while $2(P'Au, Au)$ reads $(PAu, Au) + (Au, PAu)$ i.e.

$$
(APu, Au) + (Au, APu) + ([P, A]u, Au) + (Au, [P, A]u)
$$

where

$$
|(APu, Au)| \lesssim ||Pu|| \, ||u||
$$

$$\|P''u\|^2_{(-\frac{1}{2})} = \|\langle D\rangle^{-1/2}P''u\|^2 = (\underbrace{P''}_{=(P-P')/i} u, \underbrace{\langle D\rangle^{-1}P''}_{=:A\in Op(S^0)} u)$$

$$\leq \|Pu\| \underbrace{\|Au\|}_{\lesssim \|u\|} + \left|(P'u, Au)\right|$$

Then, by Cauchy-Schwarz for the sesquilinear form $(P'v, w)$,

$$\left|(P'u, Au)\right| \leq \underbrace{(P'u, u)^{\frac{1}{2}}}_{(\mathrm{Re}(Pu,u))^{\frac{1}{2}}} (P'Au, Au)^{\frac{1}{2}}$$

while $2(P'Au, Au)$ reads $(PAu, Au) + (Au, PAu)$ i.e.

$$(APu, Au) + (Au, APu) + ([P, A]u, Au) + (Au, [P, A]u)$$

where

$$|(APu, Au)| \lesssim \|Pu\| \, \|u\|$$

$$\begin{aligned} \|P''u\|^2_{(-\frac{1}{2})} &= \|\langle D\rangle^{-1/2}P''u\|^2 = (\underbrace{P''}_{=(P-P')/i} u, \underbrace{\langle D\rangle^{-1}P''}_{=:A\in Op(S^0)} u) \\ &\leq \|Pu\| \underbrace{\|Au\|}_{\lesssim\|u\|} + \big|(P'u, Au)\big| \end{aligned}$$

Then, by Cauchy-Schwarz for the sesquilinear form $(P'v, w)$,

$$\big|(P'u, Au)\big| \leq \underbrace{(P'u, u)^{\frac{1}{2}}}_{(\mathrm{Re}(Pu,u))^{\frac{1}{2}}} (P'Au, Au)^{\frac{1}{2}}$$

while $2(P'Au, Au)$ reads $(PAu, Au) + (Au, PAu)$ i.e.

$$(APu, Au) + (Au, APu) + ([P, A]u, Au) + (Au, [P, A]u)$$

where

$$|(APu, Au)| \lesssim \|Pu\| \, \|u\|$$

$$||P''u||^2_{(-\frac{1}{2})} = ||\langle D\rangle^{-1/2}P''u||^2 = (\underbrace{P''}_{=(P-P')/i} u, \underbrace{\langle D\rangle^{-1}P''}_{=:A\in Op(S^0)} u)$$

$$\leq ||Pu|| \underbrace{||Au||}_{\lesssim ||u||} + \big|(P'u, Au)\big|$$

Then, by Cauchy-Schwarz for the sesquilinear form $(P'v, w)$,

$$\big|(P'u, Au)\big| \leq \underbrace{(P'u, u)^{\frac{1}{2}}}_{(\mathrm{Re}(Pu,u))^{\frac{1}{2}}} (P'Au, Au)^{\frac{1}{2}}$$

while $2(P'Au, Au)$ reads $(PAu, Au) + (Au, PAu)$ i.e.

$$(APu, Au) + (Au, APu) + ([P, A]u, Au) + (Au, [P, A]u)$$

where

$$|(APu, Au)| \lesssim ||Pu||\ ||u||$$

$$\begin{aligned} ||P''u||^2_{(-\frac{1}{2})} &= ||\langle D\rangle^{-1/2}P''u||^2 = (\underbrace{P''}_{=(P-P')/i} u, \underbrace{\langle D\rangle^{-1}P''}_{=:A\in Op(S^0)} u) \\ &\leq ||Pu|| \underbrace{||Au||}_{\lesssim ||u||} + \left|(P'u, Au)\right| \end{aligned}$$

Then, by Cauchy-Schwarz for the sesquilinear form $(P'v, w)$,

$$\left|(P'u, Au)\right| \leq \underbrace{(P'u, u)^{\frac{1}{2}}}_{(\mathrm{Re}(Pu,u))^{\frac{1}{2}}} (P'Au, Au)^{\frac{1}{2}}$$

while $2(P'Au, Au)$ reads $(PAu, Au) + (Au, PAu)$ i.e.

$$(APu, Au) + (Au, APu) + ([P, A]u, Au) + (Au, [P, A]u)$$

where

$$|(APu, Au)| \lesssim ||Pu||\, ||u||$$

$$
\begin{aligned}
||P''u||^2_{(-\frac{1}{2})} &= ||\langle D\rangle^{-1/2}P''u||^2 = \Big(\underbrace{P''}_{=(P-P')/i}\ u,\ \underbrace{\langle D\rangle^{-1}P''}_{=:A\in Op(S^0)}\ u\Big) \\
&\leq ||Pu||\,\underbrace{||Au||}_{\lesssim ||u||} + \big|(P'u, Au)\big|
\end{aligned}
$$

Then, by Cauchy-Schwarz for the sesquilinear form $(P'v, w)$,

$$
\big|(P'u, Au)\big| \leq \underbrace{(P'u, u)^{\frac{1}{2}}}_{(\mathrm{Re}(Pu,u))^{\frac{1}{2}}}(P'Au, Au)^{\frac{1}{2}}
$$

while $2(P'Au, Au)$ reads $(PAu, Au) + (Au, PAu)$ i.e.

$$
(APu, Au) + (Au, APu) + ([P, A]u, Au) + (Au, [P, A]u)
$$

where

$$
|(APu, Au)| \lesssim ||Pu||\ ||u||
$$

$$
\begin{aligned}
||P''u||^2_{(-\frac{1}{2})} &= ||\langle D\rangle^{-1/2}P''u||^2 = (\underbrace{P''}_{=(P-P')/i} u, \underbrace{\langle D\rangle^{-1}P''}_{=:A\in Op(S^0)} u) \\
&\leq ||Pu|| \underbrace{||Au||}_{\lesssim ||u||} + \left|(P'u, Au)\right|
\end{aligned}
$$

Then, by Cauchy-Schwarz for the sesquilinear form $(P'v, w)$,

$$
\left|(P'u, Au)\right| \leq \underbrace{(P'u, u)^{\frac{1}{2}}}_{(\mathrm{Re}(Pu,u))^{\frac{1}{2}}} (P'Au, Au)^{\frac{1}{2}}
$$

while $2(P'Au, Au)$ reads $(PAu, Au) + (Au, PAu)$ i.e.

$$
(APu, Au) + (Au, APu) + ([P, A]u, Au) + (Au, [P, A]u)
$$

where

$$
|(APu, Au)| \lesssim ||Pu|| \, ||u||
$$

$$
\begin{aligned}
||P''u||^2_{(-\frac{1}{2})} &= ||\langle D\rangle^{-1/2}P''u||^2 = (\underbrace{P''}_{=(P-P')/i} u, \underbrace{\langle D\rangle^{-1}P''}_{=:A\in Op(S^0)} u) \\
&\leq ||Pu|| \underbrace{||Au||}_{\lesssim ||u||} + \left|(P'u, Au)\right|
\end{aligned}
$$

Then, by Cauchy-Schwarz for the sesquilinear form $(P'v, w)$,

$$
\left|(P'u, Au)\right| \leq \underbrace{(P'u, u)^{\frac{1}{2}}}_{(\mathrm{Re}(Pu,u))^{\frac{1}{2}}} (P'Au, Au)^{\frac{1}{2}}
$$

while $2(P'Au, Au)$ reads $(PAu, Au) + (Au, PAu)$ i.e.

$$
(APu, Au) + (Au, APu) + ([P, A]u, Au) + (Au, [P, A]u)
$$

where

$$
|(APu, Au)| \lesssim ||Pu|| \, ||u||
$$

Observing that

$$[P,A] = \sum_1^r \underbrace{[L_j^*,A]}_{Op(S^0)} L_j + L_j^* \underbrace{[L_j,A]}_{Op(S^0)} + \underbrace{[\tilde{L}_0,A]}_{Op(S^0)} + \underbrace{[V,A]}_{Op(S^{-1})}$$

we get

$$\begin{aligned}
([P,A]u,Au) &= \sum_1^r \left([L_j^*,A]L_ju,Au\right) + \left([L_j,A]u,L_jAu\right) + O(||u||^2) \\
&= \sum_1^r \left([L_j^*,A]L_ju,Au\right) + \left([L_j,A]u,AL_ju\right) \\
&\quad + \underbrace{||[L_j,A]u||^2}_{\lesssim ||u||^2} + O(||u||^2) \\
&\lesssim ||Pu||^2 + ||u||^2.
\end{aligned}$$

This ends Step 2.

Observing that

$$[P,A] = \sum_1^r \underbrace{[L_j^*,A]}_{Op(S^0)} L_j + L_j^* \underbrace{[L_j,A]}_{Op(S^0)} + \underbrace{[\tilde{L}_0,A]}_{Op(S^0)} + \underbrace{[V,A]}_{Op(S^{-1})}$$

we get

$$\begin{aligned}
([P,A]u, Au) &= \sum_1^r ([L_j^*,A]L_ju, Au) + ([L_j,A]u, L_jAu) + O(||u||^2) \\
&= \sum_1^r ([L_j^*,A]L_ju, Au) + ([L_j,A]u, AL_ju) \\
&\quad + \underbrace{||[L_j,A]u||^2}_{\lesssim ||u||^2} + O(||u||^2) \\
&\lesssim ||Pu||^2 + ||u||^2.
\end{aligned}$$

This ends Step 2.

Observing that

$$[P,A] = \sum_1^r \underbrace{[L_j^*,A]}_{Op(S^0)} L_j + L_j^* \underbrace{[L_j,A]}_{Op(S^0)} + \underbrace{[\tilde{L}_0,A]}_{Op(S^0)} + \underbrace{[V,A]}_{Op(S^{-1})}$$

we get

$$\begin{aligned}
([P,A]u,Au) &= \sum_1^r \big([L_j^*,A]L_ju,Au\big) + \big([L_j,A]u,L_jAu\big) + O(||u||^2) \\
&= \sum_1^r \big([L_j^*,A]L_ju,Au\big) + \big([L_j,A]u,AL_ju\big) \\
&\quad + \underbrace{||[L_j,A]u||^2}_{\lesssim ||u||^2} + O(||u||^2) \\
&\lesssim ||Pu||^2 + ||u||^2.
\end{aligned}$$

This ends Step 2.

Observing that

$$[P, A] = \sum_1^r \underbrace{[L_j^*, A]}_{Op(S^0)} L_j + L_j^* \underbrace{[L_j, A]}_{Op(S^0)} + \underbrace{[\tilde{L}_0, A]}_{Op(S^0)} + \underbrace{[V, A]}_{Op(S^{-1})}$$

we get

$$\begin{aligned}
([P, A]u, Au) &= \sum_1^r \big([L_j^*, A]L_j u, Au\big) + \big([L_j, A]u, L_j Au\big) + O(||u||^2) \\
&= \sum_1^r \big([L_j^*, A]L_j u, Au\big) + \big([L_j, A]u, AL_j u\big) \\
&\quad + \underbrace{||[L_j, A]u||^2}_{\lesssim ||u||^2} + O(||u||^2) \\
&\lesssim ||Pu||^2 + ||u||^2.
\end{aligned}$$

This ends Step 2.

Observing that

$$[P,A] = \sum_1^r \underbrace{[L_j^*,A]}_{Op(S^0)} L_j + L_j^* \underbrace{[L_j,A]}_{Op(S^0)} + \underbrace{[\tilde{L}_0,A]}_{Op(S^0)} + \underbrace{[V,A]}_{Op(S^{-1})}$$

we get

$$\begin{aligned}([P,A]u,Au) &= \sum_1^r \big([L_j^*,A]L_ju,Au\big) + \big([L_j,A]u,L_jAu\big) + O(||u||^2)\\ &= \sum_1^r \big([L_j^*,A]L_ju,Au\big) + \big([L_j,A]u,AL_ju\big)\\ &\quad + \underbrace{||[L_j,A]u||^2}_{\lesssim ||u||^2} + O(||u||^2)\\ &\lesssim ||Pu||^2 + ||u||^2.\end{aligned}$$

This ends Step 2.

Observing that

$$[P,A] = \sum_{1}^{r} \underbrace{[L_j^*,A]}_{Op(S^0)} L_j + L_j^* \underbrace{[L_j,A]}_{Op(S^0)} + \underbrace{[\tilde{L}_0,A]}_{Op(S^0)} + \underbrace{[V,A]}_{Op(S^{-1})}$$

we get

$$\begin{aligned} ([P,A]u,Au) &= \sum_{1}^{r} \big([L_j^*,A]L_j u, Au\big) + \big([L_j,A]u, L_j Au\big) + O(||u||^2) \\ &= \sum_{1}^{r} \big([L_j^*,A]L_j u, Au\big) + \big([L_j,A]u, AL_j u\big) \\ &\quad + \underbrace{||[L_j,A]u||^2}_{\lesssim ||u||^2} + O(||u||^2) \\ &\lesssim ||Pu||^2 + ||u||^2. \end{aligned}$$

This ends Step 2.

Observing that

$$[P,A]=\sum_1^r \underbrace{[L_j^*,A]}_{Op(S^0)} L_j + L_j^* \underbrace{[L_j,A]}_{Op(S^0)} + \underbrace{[\tilde{L}_0,A]}_{Op(S^0)} + \underbrace{[V,A]}_{Op(S^{-1})}$$

we get

$$\begin{aligned}([P,A]u,Au) &= \sum_1^r \big([L_j^*,A]L_ju,Au\big) + \big([L_j,A]u,L_jAu\big) + O(||u||^2) \\ &= \sum_1^r \big([L_j^*,A]L_ju,Au\big) + \big([L_j,A]u,AL_ju\big) \\ &\quad + \underbrace{||[L_j,A]u||^2}_{\lesssim ||u||^2} + O(||u||^2) \\ &\lesssim ||Pu||^2 + ||u||^2.\end{aligned}$$

This ends Step 2.

Observing that

$$[P, A] = \sum_1^r \underbrace{[L_j^*, A]}_{Op(S^0)} L_j + L_j^* \underbrace{[L_j, A]}_{Op(S^0)} + \underbrace{[\tilde{L}_0, A]}_{Op(S^0)} + \underbrace{[V, A]}_{Op(S^{-1})}$$

we get

$$\begin{aligned}
([P, A]u, Au) &= \sum_1^r \big([L_j^*, A]L_j u, Au\big) + \big([L_j, A]u, L_j Au\big) + O(||u||^2) \\
&= \sum_1^r \big([L_j^*, A]L_j u, Au\big) + \big([L_j, A]u, AL_j u\big) \\
&\quad + \underbrace{||[L_j, A]u||^2}_{\lesssim ||u||^2} + O(||u||^2) \\
&\lesssim ||Pu||^2 + ||u||^2.
\end{aligned}$$

This ends Step 2.

Observing that

$$[P, A] = \sum_1^r \underbrace{[L_j^*, A]}_{Op(S^0)} L_j + L_j^* \underbrace{[L_j, A]}_{Op(S^0)} + \underbrace{[\tilde{L}_0, A]}_{Op(S^0)} + \underbrace{[V, A]}_{Op(S^{-1})}$$

we get

$$\begin{aligned}
([P, A]u, Au) &= \sum_1^r \big([L_j^*, A]L_j u, Au\big) + \big([L_j, A]u, L_j A u\big) + O(||u||^2) \\
&= \sum_1^r \big([L_j^*, A]L_j u, Au\big) + \big([L_j, A]u, A L_j u\big) \\
&\quad + \underbrace{||[L_j, A]u||^2}_{\lesssim ||u||^2} + O(||u||^2) \\
&\lesssim ||Pu||^2 + ||u||^2.
\end{aligned}$$

This ends Step 2.

Observing that

$$[P, A] = \sum_1^r \underbrace{[L_j^*, A]}_{Op(S^0)} L_j + L_j^* \underbrace{[L_j, A]}_{Op(S^0)} + \underbrace{[\tilde{L}_0, A]}_{Op(S^0)} + \underbrace{[V, A]}_{Op(S^{-1})}$$

we get

$$\begin{aligned}
([P, A]u, Au) &= \sum_1^r \big([L_j^*, A]L_j u, Au\big) + \big([L_j, A]u, L_j Au\big) + O(||u||^2) \\
&= \sum_1^r \big([L_j^*, A]L_j u, Au\big) + \big([L_j, A]u, AL_j u\big) \\
&\quad + \underbrace{||[L_j, A]u||^2}_{\lesssim ||u||^2} + O(||u||^2) \\
&\lesssim ||Pu||^2 + ||u||^2.
\end{aligned}$$

This ends Step 2.

Observing that

$$[P,A] = \sum_1^r \underbrace{[L_j^*,A]}_{Op(S^0)} L_j + L_j^* \underbrace{[L_j,A]}_{Op(S^0)} + \underbrace{[\tilde{L}_0,A]}_{Op(S^0)} + \underbrace{[V,A]}_{Op(S^{-1})}$$

we get

$$\begin{aligned}([P,A]u,Au) &= \sum_1^r \big([L_j^*,A]L_ju,Au\big) + \big([L_j,A]u,L_jAu\big) + O(||u||^2) \\ &= \sum_1^r \big([L_j^*,A]L_ju,Au\big) + \big([L_j,A]u,AL_ju\big) \\ &\quad + \underbrace{||[L_j,A]u||^2}_{\lesssim ||u||^2} + O(||u||^2) \\ &\lesssim ||Pu||^2 + ||u||^2.\end{aligned}$$

This ends Step 2.

Observing that

$$[P,A]=\sum_1^r \underbrace{[L_j^*,A]}_{Op(S^0)}L_j+L_j^*\underbrace{[L_j,A]}_{Op(S^0)}+\underbrace{[\tilde{L}_0,A]}_{Op(S^0)}+\underbrace{[V,A]}_{Op(S^{-1})}$$

we get

$$\begin{aligned}([P,A]u,Au) &= \sum_1^r \big([L_j^*,A]L_ju,Au\big)+\big([L_j,A]u,L_jAu\big)+O(||u||^2)\\ &= \sum_1^r \big([L_j^*,A]L_ju,Au\big)+\big([L_j,A]u,AL_ju\big)\\ &\quad +\underbrace{||[L_j,A]u||^2}_{\lesssim ||u||^2}+O(||u||^2)\\ &\lesssim ||Pu||^2+||u||^2.\end{aligned}$$

This ends Step 2.

Observing that

$$[P,A] = \sum_1^r \underbrace{[L_j^*, A]}_{Op(S^0)} L_j + L_j^* \underbrace{[L_j, A]}_{Op(S^0)} + \underbrace{[\tilde{L}_0, A]}_{Op(S^0)} + \underbrace{[V, A]}_{Op(S^{-1})}$$

we get

$$\begin{aligned}
([P,A]u, Au) &= \sum_1^r \big([L_j^*, A]L_j u, Au\big) + \big([L_j, A]u, L_j Au\big) + O(||u||^2) \\
&= \sum_1^r \big([L_j^*, A]L_j u, Au\big) + \big([L_j, A]u, AL_j u\big) \\
&\quad + \underbrace{||[L_j, A]u||^2}_{\lesssim ||u||^2} + O(||u||^2) \\
&\lesssim ||Pu||^2 + ||u||^2.
\end{aligned}$$

This ends Step 2.

Observing that

$$[P,A] = \sum_1^r \underbrace{[L_j^*, A]}_{Op(S^0)} L_j + L_j^* \underbrace{[L_j, A]}_{Op(S^0)} + \underbrace{[\tilde{L}_0, A]}_{Op(S^0)} + \underbrace{[V, A]}_{Op(S^{-1})}$$

we get

$$\begin{aligned}
([P,A]u, Au) &= \sum_1^r \big([L_j^*, A]L_j u, Au\big) + \big([L_j, A]u, L_j Au\big) + O(||u||^2) \\
&= \sum_1^r \big([L_j^*, A]L_j u, Au\big) + \big([L_j, A]u, AL_j u\big) \\
&\quad + \underbrace{||[L_j, A]u||^2}_{\lesssim ||u||^2} + O(||u||^2) \\
&\lesssim ||Pu||^2 + ||u||^2.
\end{aligned}$$

This ends Step 2.

Proof: the general induction step

We let $s_k = 2^{1-k} - 1$ for $k \geq 1$ ($s_1 = 0$, $s_2 = -\frac{1}{2}$). We have seen

$$\begin{aligned} ||L_j u||_{(s_1)} &\lesssim ||Pu|| + ||u|| \qquad (1 \leq j \leq r) \\ ||\tilde{L}_0 u||_{(s_2)} &\lesssim ||Pu|| + ||u|| \end{aligned}$$

One defines next

$$\mathcal{Q}_1 = \{iL_j \mid j = 1, \ldots, r\} \qquad \mathcal{Q}_2 = \underbrace{\{i\tilde{L}_0\}}_{\text{or eq. } \{P''\}} \cup \{i[Q, Q'] | Q, Q' \in \mathcal{Q}_1\}$$

and, for $k \geq 2$,

$$\mathcal{Q}_{k+1} = \{i[Q, Q'] \mid (Q, Q') \in (\mathcal{Q}_k \times \mathcal{Q}_1) \cup (\mathcal{Q}_{k-1} \times \mathcal{Q}_2)\}$$

Remark. For each $Q_k \in \mathcal{Q}_k$, $Q_k^* - Q_k \in Op(S^0)$.

Induction assumption: $||Q_k u||_{(s_k)} \lesssim ||Pu|| + ||u||$

Proof: the general induction step

We let $s_k = 2^{1-k} - 1$ for $k \geq 1$ ($s_1 = 0$, $s_2 = -\frac{1}{2}$). We have seen

$$||L_j u||_{(s_1)} \lesssim ||Pu|| + ||u|| \qquad (1 \leq j \leq r)$$
$$||\tilde{L}_0 u||_{(s_2)} \lesssim ||Pu|| + ||u||$$

One defines next

$$\mathcal{Q}_1 = \{iL_j \mid j = 1, \ldots, r\} \qquad \mathcal{Q}_2 = \underbrace{\{i\tilde{L}_0\}}_{\text{or eq. } \{P''\}} \cup \{i[Q, Q'] | Q, Q' \in \mathcal{Q}_1\}$$

and, for $k \geq 2$,

$$\mathcal{Q}_{k+1} = \{i[Q, Q'] \mid (Q, Q') \in (\mathcal{Q}_k \times \mathcal{Q}_1) \cup (\mathcal{Q}_{k-1} \times \mathcal{Q}_2)\}$$

Remark. For each $Q_k \in \mathcal{Q}_k$, $Q_k^* - Q_k \in Op(S^0)$.

Induction assumption: $||Q_k u||_{(s_k)} \lesssim ||Pu|| + ||u||$

Proof: the general induction step

We let $s_k = 2^{1-k} - 1$ for $k \geq 1$ ($s_1 = 0$, $s_2 = -\frac{1}{2}$). We have seen

$$
\begin{aligned}
||L_j u||_{(s_1)} &\lesssim ||Pu|| + ||u|| \qquad (1 \leq j \leq r) \\
||\tilde{L}_0 u||_{(s_2)} &\lesssim ||Pu|| + ||u||
\end{aligned}
$$

One defines next

$$
\mathcal{Q}_1 = \{iL_j \mid j = 1, \ldots, r\} \qquad \mathcal{Q}_2 = \underbrace{\{i\tilde{L}_0\}}_{\text{or eq. } \{P''\}} \cup \{i[Q, Q'] | Q, Q' \in \mathcal{Q}_1\}
$$

and, for $k \geq 2$,

$$
\mathcal{Q}_{k+1} = \{i[Q, Q'] \mid (Q, Q') \in (\mathcal{Q}_k \times \mathcal{Q}_1) \cup (\mathcal{Q}_{k-1} \times \mathcal{Q}_2)\}
$$

Remark. For each $Q_k \in \mathcal{Q}_k$, $Q_k^* - Q_k \in Op(S^0)$.

Induction assumption: $||Q_k u||_{(s_k)} \lesssim ||Pu|| + ||u||$

Proof: the general induction step

We let $s_k = 2^{1-k} - 1$ for $k \geq 1$ ($s_1 = 0$, $s_2 = -\frac{1}{2}$). We have seen

$$\begin{aligned} ||L_j u||_{(s_1)} &\lesssim ||Pu|| + ||u|| \qquad (1 \leq j \leq r) \\ ||\tilde{L}_0 u||_{(s_2)} &\lesssim ||Pu|| + ||u|| \end{aligned}$$

One defines next

$$\mathcal{Q}_1 = \{iL_j \mid j = 1, \ldots, r\} \qquad \mathcal{Q}_2 = \underbrace{\{i\tilde{L}_0\}}_{\text{or eq. } \{P''\}} \cup \{i[Q, Q'] | Q, Q' \in \mathcal{Q}_1\}$$

and, for $k \geq 2$,

$$\mathcal{Q}_{k+1} = \{i[Q, Q'] \mid (Q, Q') \in (\mathcal{Q}_k \times \mathcal{Q}_1) \cup (\mathcal{Q}_{k-1} \times \mathcal{Q}_2)\}$$

Remark. For each $Q_k \in \mathcal{Q}_k$, $Q_k^* - Q_k \in Op(S^0)$.

Induction assumption: $||Q_k u||_{(s_k)} \lesssim ||Pu|| + ||u||$

Proof: the general induction step

We let $s_k = 2^{1-k} - 1$ for $k \geq 1$ ($s_1 = 0$, $s_2 = -\frac{1}{2}$). We have seen

$$\begin{aligned} ||L_j u||_{(s_1)} &\lesssim ||Pu|| + ||u|| \qquad (1 \leq j \leq r) \\ ||\tilde{L}_0 u||_{(s_2)} &\lesssim ||Pu|| + ||u|| \end{aligned}$$

One defines next

$$\mathcal{Q}_1 = \{iL_j \mid j = 1, \ldots, r\} \qquad \mathcal{Q}_2 = \underbrace{\{i\tilde{L}_0\}}_{\text{or eq. } \{P''\}} \cup \{i[Q, Q'] | Q, Q' \in \mathcal{Q}_1\}$$

and, for $k \geq 2$,

$$\mathcal{Q}_{k+1} = \{i[Q, Q'] \mid (Q, Q') \in (\mathcal{Q}_k \times \mathcal{Q}_1) \cup (\mathcal{Q}_{k-1} \times \mathcal{Q}_2)\}$$

Remark. For each $Q_k \in \mathcal{Q}_k$, $Q_k^* - Q_k \in Op(S^0)$.

Induction assumption: $||Q_k u||_{(s_k)} \lesssim ||Pu|| + ||u||$

Proof: the general induction step

We let $s_k = 2^{1-k} - 1$ for $k \geq 1$ ($s_1 = 0$, $s_2 = -\frac{1}{2}$). We have seen

$$\begin{aligned} ||L_j u||_{(s_1)} &\lesssim ||Pu|| + ||u|| \qquad (1 \leq j \leq r) \\ ||\tilde{L}_0 u||_{(s_2)} &\lesssim ||Pu|| + ||u|| \end{aligned}$$

One defines next

$$\mathcal{Q}_1 = \{iL_j \mid j = 1, \ldots, r\} \qquad \mathcal{Q}_2 = \underbrace{\{i\tilde{L}_0\}}_{\text{or eq. } \{P''\}} \cup \{i[Q, Q'] | Q, Q' \in \mathcal{Q}_1\}$$

and, for $k \geq 2$,

$$\mathcal{Q}_{k+1} = \{i[Q, Q'] \mid (Q, Q') \in (\mathcal{Q}_k \times \mathcal{Q}_1) \cup (\mathcal{Q}_{k-1} \times \mathcal{Q}_2)\}$$

Remark. For each $Q_k \in \mathcal{Q}_k$, $Q_k^* - Q_k \in Op(S^0)$.

Induction assumption: $||Q_k u||_{(s_k)} \lesssim ||Pu|| + ||u||$

Proof: estimate on $i[Q, Q']$ with $Q \in \mathcal{Q}_k$ and $Q' \in \mathcal{Q}_1$

$$
\begin{aligned}
\|[Q_1, Q_k]u\|^2_{(s_{k+1})} &= (\langle D\rangle^{s_{k+1}}[Q_1, Q_k]u, \langle D\rangle^{s_{k+1}}[Q_1, Q_k]u) \\
&= ([Q_1, Q_k]u, \underbrace{\langle D\rangle^{2s_{k+1}}[Q_1, Q_k]}_{=:A \in OpS^{1+2s_{k+1}} = Op(S^{s_k})} u) \\
&= (Q_k u, Q_1^* Au) - (Q_1 u, Q_k^* Au) \\
&= (Q_k u, AQ_1 u) + (Q_k u, \underbrace{((Q_1^* - Q_1)A + [Q_1, A])}_{B \in Op(S^{s_k})} u) \\
&- (Q_1 u, AQ_k u) - (Q_1 u, \underbrace{((Q_k^* - Q_k)A + [Q_k, A])}_{C \in Op(S^{s_k}) \subset Op(S^0)} u) \\
&= (A^* Q_k u, Q_1 u) + (B^* Q_k u, u) \\
&\quad - (Q_1 u, AQ_k u) - (Q_1 u, Cu) \\
&= O(\|Q_1 u\|_{(s_1)}\|Q_k u\|_{(s_k)} + \|Q_1 u\|_{(s_1)}\|u\| \\
&\quad + \|Q_k u\|_{(s_k)}\|u\|)
\end{aligned}
$$

Proof: estimate on $i[Q, Q']$ with $Q \in \mathcal{Q}_k$ and $Q' \in \mathcal{Q}_1$

$$
\begin{aligned}
\left|\left|[Q_1, Q_k]u\right|\right|^2_{(s_{k+1})} &= (\langle D\rangle^{s_{k+1}}[Q_1, Q_k]u, \langle D\rangle^{s_{k+1}}[Q_1, Q_k]u) \\
&= ([Q_1, Q_k]u, \underbrace{\langle D\rangle^{2s_{k+1}}[Q_1, Q_k]}_{=:A \in OpS^{1+2s_{k+1}} = Op(S^{s_k})} u) \\
&= (Q_k u, Q_1^* Au) - (Q_1 u, Q_k^* Au) \\
&= (Q_k u, AQ_1 u) + (Q_k u, \underbrace{((Q_1^* - Q_1)A + [Q_1, A])}_{B \in Op(S^{s_k})} u) \\
&- (Q_1 u, AQ_k u) - (Q_1 u, \underbrace{((Q_k^* - Q_k)A + [Q_k, A])}_{C \in Op(S^{s_k}) \subset Op(S^0)} u) \\
&= (A^* Q_k u, Q_1 u) + (B^* Q_k u, u) \\
&\quad - (Q_1 u, AQ_k u) - (Q_1 u, Cu) \\
&= O(||Q_1 u||_{(s_1)}||Q_k u||_{(s_k)} + ||Q_1 u||_{(s_1)}||u|| \\
&\quad + ||Q_k u||_{(s_k)}||u||)
\end{aligned}
$$

Proof: estimate on $i[Q, Q']$ with $Q \in \mathcal{Q}_k$ and $Q' \in \mathcal{Q}_1$

$$
\begin{aligned}
\big|\big|[Q_1, Q_k]u\big|\big|^2_{(s_{k+1})} &= (\langle D\rangle^{s_{k+1}}[Q_1, Q_k]u, \langle D\rangle^{s_{k+1}}[Q_1, Q_k]u) \\
&= ([Q_1, Q_k]u, \underbrace{\langle D\rangle^{2s_{k+1}}[Q_1, Q_k]}_{=:A \in OpS^{1+2s_{k+1}} = Op(S^{s_k})} u) \\
&= (Q_k u, Q_1^* A u) - (Q_1 u, Q_k^* A u) \\
&= (Q_k u, A Q_1 u) + (Q_k u, \underbrace{((Q_1^* - Q_1)A + [Q_1, A])}_{B \in Op(S^{s_k})} u \\
&- \ (Q_1 u, A Q_k u) - (Q_1 u, \underbrace{((Q_k^* - Q_k)A + [Q_k, A])}_{C \in Op(S^{s_k}) \subset Op(S^0)} u \\
&= (A^* Q_k u, Q_1 u) + (B^* Q_k u, u) \\
&\quad - (Q_1 u, A Q_k u) - (Q_1 u, C u) \\
&= O(||Q_1 u||_{(s_1)} ||Q_k u||_{(s_k)} + ||Q_1 u||_{(s_1)} ||u|| \\
&\quad + ||Q_k u||_{(s_k)} ||u||)
\end{aligned}
$$

Proof: estimate on $i[Q, Q']$ with $Q \in \mathcal{Q}_k$ and $Q' \in \mathcal{Q}_1$

$$
\begin{aligned}
\left|\left|[Q_1, Q_k]u\right|\right|^2_{(s_{k+1})} &= (\langle D\rangle^{s_{k+1}}[Q_1, Q_k]u, \langle D\rangle^{s_{k+1}}[Q_1, Q_k]u) \\
&= ([Q_1, Q_k]u, \underbrace{\langle D\rangle^{2s_{k+1}}[Q_1, Q_k]}_{=:A\in OpS^{1+2s_{k+1}}=Op(S^{s_k})} u) \\
&= (Q_k u, Q_1^* Au) - (Q_1 u, Q_k^* Au) \\
&= (Q_k u, AQ_1 u) + (Q_k u, \underbrace{((Q_1^* - Q_1)A + [Q_1, A])}_{B\in Op(S^{s_k})} u) \\
&- (Q_1 u, AQ_k u) - (Q_1 u, \underbrace{((Q_k^* - Q_k)A + [Q_k, A])}_{C\in Op(S^{s_k})\subset Op(S^0)} u) \\
&= (A^* Q_k u, Q_1 u) + (B^* Q_k u, u) \\
&\quad - (Q_1 u, AQ_k u) - (Q_1 u, Cu) \\
&= O(||Q_1 u||_{(s_1)}||Q_k u||_{(s_k)} + ||Q_1 u||_{(s_1)}||u|| \\
&\quad + ||Q_k u||_{(s_k)}||u||)
\end{aligned}
$$

Proof: estimate on $i[Q, Q']$ with $Q \in \mathcal{Q}_k$ and $Q' \in \mathcal{Q}_1$

$$
\begin{aligned}
\big|\big|[Q_1, Q_k]u\big|\big|^2_{(s_{k+1})} &= (\langle D\rangle^{s_{k+1}}[Q_1, Q_k]u, \langle D\rangle^{s_{k+1}}[Q_1, Q_k]u) \\
&= \big([Q_1, Q_k]u, \underbrace{\langle D\rangle^{2s_{k+1}}[Q_1, Q_k]}_{=:A\in OpS^{1+2s_{k+1}}=Op(S^{s_k})} u\big) \\
&= (Q_k u, Q_1^* Au) - (Q_1 u, Q_k^* Au) \\
&= (Q_k u, AQ_1 u) + \big(Q_k u, \underbrace{((Q_1^* - Q_1)A + [Q_1, A])}_{B\in Op(S^{s_k})} u\big) \\
&- (Q_1 u, AQ_k u) - \big(Q_1 u, \underbrace{((Q_k^* - Q_k)A + [Q_k, A])}_{C\in Op(S^{s_k})\subset Op(S^0)} u\big) \\
&= (A^* Q_k u, Q_1 u) + (B^* Q_k u, u) \\
&\quad - (Q_1 u, AQ_k u) - (Q_1 u, Cu) \\
&= O\big(||Q_1 u||_{(s_1)}||Q_k u||_{(s_k)} + ||Q_1 u||_{(s_1)}||u|| \\
&\quad + ||Q_k u||_{(s_k)}||u||\big)
\end{aligned}
$$

Proof: estimate on $i[Q, Q']$ with $Q \in \mathcal{Q}_k$ and $Q' \in \mathcal{Q}_1$

$$
\begin{aligned}
\left|\left|[Q_1, Q_k]u\right|\right|^2_{(s_{k+1})} &= (\langle D\rangle^{s_{k+1}}[Q_1, Q_k]u, \langle D\rangle^{s_{k+1}}[Q_1, Q_k]u) \\
&= \Big([Q_1, Q_k]u, \underbrace{\langle D\rangle^{2s_{k+1}}[Q_1, Q_k]}_{=:A\in OpS^{1+2s_{k+1}}=Op(S^{s_k})} u\Big) \\
&= (Q_k u, Q_1^* A u) - (Q_1 u, Q_k^* A u) \\
&= (Q_k u, AQ_1 u) + \Big(Q_k u, \underbrace{((Q_1^* - Q_1)A + [Q_1, A])}_{B\in Op(S^{s_k})} u\Big) \\
&- (Q_1 u, AQ_k u) - \Big(Q_1 u, \underbrace{((Q_k^* - Q_k)A + [Q_k, A])}_{C\in Op(S^{s_k})\subset Op(S^0)} u\Big) \\
&= (A^* Q_k u, Q_1 u) + (B^* Q_k u, u) \\
&\quad - (Q_1 u, AQ_k u) - (Q_1 u, Cu) \\
&= O\big(||Q_1 u||_{(s_1)}||Q_k u||_{(s_k)} + ||Q_1 u||_{(s_1)}||u|| \\
&\quad + ||Q_k u||_{(s_k)}||u||\big)
\end{aligned}
$$

Proof: estimate on $i[Q, Q']$ with $Q \in \mathcal{Q}_k$ and $Q' \in \mathcal{Q}_1$

$$\left|\left|[Q_1, Q_k]u\right|\right|^2_{(s_{k+1})} = (\langle D\rangle^{s_{k+1}}[Q_1, Q_k]u, \langle D\rangle^{s_{k+1}}[Q_1, Q_k]u)$$

$$= \Big([Q_1, Q_k]u, \quad \underbrace{\langle D\rangle^{2s_{k+1}}[Q_1, Q_k]}_{=:A\in OpS^{1+2s_{k+1}} = Op(S^{s_k})} \quad u\Big)$$

$$= (Q_k u, Q_1^* Au) - (Q_1 u, Q_k^* Au)$$

$$= (Q_k u, AQ_1 u) + \Big(Q_k u, \underbrace{((Q_1^* - Q_1)A + [Q_1, A])}_{B\in Op(S^{s_k})} u$$

$$- (Q_1 u, AQ_k u) - \Big(Q_1 u, \underbrace{((Q_k^* - Q_k)A + [Q_k, A])}_{C\in Op(S^{s_k})\subset Op(S^0)} u$$

$$= (A^* Q_k u, Q_1 u) + (B^* Q_k u, u)$$

$$- (Q_1 u, AQ_k u) - (Q_1 u, Cu)$$

$$= O\Big(||Q_1 u||_{(s_1)}||Q_k u||_{(s_k)} + ||Q_1 u||_{(s_1)}||u||$$

$$+ ||Q_k u||_{(s_k)}||u||\Big)$$

Proof: estimate on $i[Q, Q']$ with $Q \in \mathcal{Q}_k$ and $Q' \in \mathcal{Q}_1$

$$
\begin{aligned}
||[Q_1, Q_k]u||^2_{(s_{k+1})} &= (\langle D\rangle^{s_{k+1}}[Q_1, Q_k]u, \langle D\rangle^{s_{k+1}}[Q_1, Q_k]u) \\
&= \Big([Q_1, Q_k]u, \underbrace{\langle D\rangle^{2s_{k+1}}[Q_1, Q_k]}_{=:A\in OpS^{1+2s_{k+1}} = Op(S^{s_k})} u\Big) \\
&= (Q_k u, Q_1^* Au) - (Q_1 u, Q_k^* Au) \\
&= (Q_k u, AQ_1 u) + \Big(Q_k u, \underbrace{((Q_1^* - Q_1)A + [Q_1, A])}_{B\in Op(S^{s_k})} u\Big) \\
&- (Q_1 u, AQ_k u) - \Big(Q_1 u, \underbrace{((Q_k^* - Q_k)A + [Q_k, A])}_{C\in Op(S^{s_k})\subset Op(S^0)} u\Big) \\
&= (A^* Q_k u, Q_1 u) + (B^* Q_k u, u) \\
&\quad - (Q_1 u, AQ_k u) - (Q_1 u, Cu) \\
&= O(||Q_1 u||_{(s_1)}||Q_k u||_{(s_k)} + ||Q_1 u||_{(s_1)}||u|| \\
&\quad + ||Q_k u||_{(s_k)}||u||)
\end{aligned}
$$

Proof: estimate on $i[Q, Q']$ with $Q \in \mathcal{Q}_k$ and $Q' \in \mathcal{Q}_1$

$$
\begin{aligned}
\left|\left|[Q_1, Q_k]u\right|\right|^2_{(s_{k+1})} &= (\langle D\rangle^{s_{k+1}}[Q_1, Q_k]u, \langle D\rangle^{s_{k+1}}[Q_1, Q_k]u) \\
&= \Big([Q_1, Q_k]u, \underbrace{\langle D\rangle^{2s_{k+1}}[Q_1, Q_k]}_{=:A\in OpS^{1+2s_{k+1}} = Op(S^{s_k})} u\Big) \\
&= (Q_k u, Q_1^* A u) - (Q_1 u, Q_k^* A u) \\
&= (Q_k u, A Q_1 u) + \Big(Q_k u, \underbrace{((Q_1^* - Q_1)A + [Q_1, A])}_{B\in Op(S^{s_k})} u\Big) \\
&- (Q_1 u, A Q_k u) - \Big(Q_1 u, \underbrace{((Q_k^* - Q_k)A + [Q_k, A])}_{C\in Op(S^{s_k})\subset Op(S^0)} u\Big) \\
&= (A^* Q_k u, Q_1 u) + (B^* Q_k u, u) \\
&\quad - (Q_1 u, A Q_k u) - (Q_1 u, C u) \\
&= O(||Q_1 u||_{(s_1)} ||Q_k u||_{(s_k)} + ||Q_1 u||_{(s_1)} ||u|| \\
&\quad + ||Q_k u||_{(s_k)} ||u||)
\end{aligned}
$$

Proof: estimate on $i[Q, Q']$ with $Q \in \mathcal{Q}_k$ and $Q' \in \mathcal{Q}_1$

$$
\begin{aligned}
\|[Q_1, Q_k]u\|^2_{(s_{k+1})} &= (\langle D\rangle^{s_{k+1}}[Q_1, Q_k]u, \langle D\rangle^{s_{k+1}}[Q_1, Q_k]u) \\
&= \big([Q_1, Q_k]u, \underbrace{\langle D\rangle^{2s_{k+1}}[Q_1, Q_k]}_{=:A\in OpS^{1+2s_{k+1}}=Op(S^{s_k})} u\big) \\
&= (Q_k u, Q_1^* A u) - (Q_1 u, Q_k^* A u) \\
&= (Q_k u, AQ_1 u) + (Q_k u, \underbrace{((Q_1^* - Q_1)A + [Q_1, A])}_{B\in Op(S^{s_k})} u) \\
&- (Q_1 u, AQ_k u) - (Q_1 u, \underbrace{((Q_k^* - Q_k)A + [Q_k, A])}_{C\in Op(S^{s_k})\subset Op(S^0)} u) \\
&= (A^* Q_k u, Q_1 u) + (B^* Q_k u, u) \\
&\quad - (Q_1 u, AQ_k u) - (Q_1 u, Cu) \\
&= O(\|Q_1 u\|_{(s_1)}\|Q_k u\|_{(s_k)} + \|Q_1 u\|_{(s_1)}\|u\| \\
&\quad + \|Q_k u\|_{(s_k)}\|u\|)
\end{aligned}
$$

Proof: estimate on $i[Q, Q']$ with $Q \in \mathcal{Q}_k$ and $Q' \in \mathcal{Q}_1$

$$
\begin{aligned}
\big|\big|[Q_1, Q_k]u\big|\big|^2_{(s_{k+1})} &= (\langle D\rangle^{s_{k+1}}[Q_1, Q_k]u, \langle D\rangle^{s_{k+1}}[Q_1, Q_k]u) \\
&= \Big([Q_1, Q_k]u, \underbrace{\langle D\rangle^{2s_{k+1}}[Q_1, Q_k]}_{=:A\in OpS^{1+2s_{k+1}} = Op(S^{s_k})} u\Big) \\
&= (Q_k u, Q_1^* A u) - (Q_1 u, Q_k^* A u) \\
&= (Q_k u, A Q_1 u) + \Big(Q_k u, \underbrace{((Q_1^* - Q_1)A + [Q_1, A])}_{B\in Op(S^{s_k})} u\Big) \\
&- (Q_1 u, A Q_k u) - \Big(Q_1 u, \underbrace{((Q_k^* - Q_k)A + [Q_k, A])}_{C\in Op(S^{s_k})\subset Op(S^0)} u\Big) \\
&= (A^* Q_k u, Q_1 u) + (B^* Q_k u, u) \\
&\quad - (Q_1 u, A Q_k u) - (Q_1 u, C u) \\
&= O\big(||Q_1 u||_{(s_1)}||Q_k u||_{(s_k)} + ||Q_1 u||_{(s_1)}||u|| \\
&\quad + ||Q_k u||_{(s_k)}||u||\big)
\end{aligned}
$$

Proof: estimate on $i[Q, Q']$ with $Q \in \mathcal{Q}_k$ and $Q' \in \mathcal{Q}_1$

$$
\begin{aligned}
\left|\left|[Q_1, Q_k]u\right|\right|^2_{(s_{k+1})} &= (\langle D\rangle^{s_{k+1}}[Q_1, Q_k]u, \langle D\rangle^{s_{k+1}}[Q_1, Q_k]u) \\
&= \Big([Q_1, Q_k]u, \underbrace{\langle D\rangle^{2s_{k+1}}[Q_1, Q_k]}_{=:A\in OpS^{1+2s_{k+1}}=Op(S^{s_k})} u\Big) \\
&= (Q_k u, Q_1^* Au) - (Q_1 u, Q_k^* Au) \\
&= (Q_k u, AQ_1 u) + \Big(Q_k u, \underbrace{((Q_1^* - Q_1)A + [Q_1, A])}_{B\in Op(S^{s_k})} u\Big) \\
&- (Q_1 u, AQ_k u) - \Big(Q_1 u, \underbrace{((Q_k^* - Q_k)A + [Q_k, A])}_{C\in Op(S^{s_k})\subset Op(S^0)} u\Big) \\
&= (A^* Q_k u, Q_1 u) + (B^* Q_k u, u) \\
&\quad - (Q_1 u, AQ_k u) - (Q_1 u, Cu) \\
&= O(||Q_1 u||_{(s_1)}||Q_k u||_{(s_k)} + ||Q_1 u||_{(s_1)}||u|| \\
&\quad + ||Q_k u||_{(s_k)}||u||)
\end{aligned}
$$

Proof: estimate on $i[Q, Q']$ with $Q \in \mathcal{Q}_k$ and $Q' \in \mathcal{Q}_1$

$$
\begin{aligned}
\left|\left|[Q_1, Q_k]u\right|\right|^2_{(s_{k+1})} &= (\langle D\rangle^{s_{k+1}}[Q_1, Q_k]u, \langle D\rangle^{s_{k+1}}[Q_1, Q_k]u) \\
&= \Big([Q_1, Q_k]u, \quad \underbrace{\langle D\rangle^{2s_{k+1}}[Q_1, Q_k]}_{=:A\in OpS^{1+2s_{k+1}}=Op(S^{s_k})} \quad u\Big) \\
&= (Q_k u, Q_1^* Au) - (Q_1 u, Q_k^* Au) \\
&= (Q_k u, AQ_1 u) + \Big(Q_k u, \underbrace{((Q_1^* - Q_1)A + [Q_1, A])}_{B\in Op(S^{s_k})} u\Big) \\
&- (Q_1 u, AQ_k u) - \Big(Q_1 u, \underbrace{((Q_k^* - Q_k)A + [Q_k, A])}_{C\in Op(S^{s_k})\subset Op(S^0)} u\Big) \\
&= (A^* Q_k u, Q_1 u) + (B^* Q_k u, u) \\
&\quad - (Q_1 u, AQ_k u) - (Q_1 u, Cu) \\
&= O(||Q_1 u||_{(s_1)}||Q_k u||_{(s_k)} + ||Q_1 u||_{(s_1)}||u|| \\
&\quad + ||Q_k u||_{(s_k)}||u||)
\end{aligned}
$$

Proof: estimate on $i[Q, Q']$ with $Q \in \mathcal{Q}_k$ and $Q' \in \mathcal{Q}_1$

$$
\begin{aligned}
||[Q_1, Q_k]u||^2_{(s_{k+1})} &= (\langle D\rangle^{s_{k+1}}[Q_1, Q_k]u, \langle D\rangle^{s_{k+1}}[Q_1, Q_k]u) \\
&= ([Q_1, Q_k]u, \underbrace{\langle D\rangle^{2s_{k+1}}[Q_1, Q_k]}_{=:A\in OpS^{1+2s_{k+1}}=Op(S^{s_k})} u) \\
&= (Q_k u, Q_1^* Au) - (Q_1 u, Q_k^* Au) \\
&= (Q_k u, AQ_1 u) + (Q_k u, \underbrace{((Q_1^* - Q_1)A + [Q_1, A])}_{B\in Op(S^{s_k})} u) \\
&- (Q_1 u, AQ_k u) - (Q_1 u, \underbrace{((Q_k^* - Q_k)A + [Q_k, A])}_{C\in Op(S^{s_k})\subset Op(S^0)} u) \\
&= (A^* Q_k u, Q_1 u) + (B^* Q_k u, u) \\
&\quad - (Q_1 u, AQ_k u) - (Q_1 u, Cu) \\
&= O(||Q_1 u||_{(s_1)}||Q_k u||_{(s_k)} + ||Q_1 u||_{(s_1)}||u|| \\
&\quad + ||Q_k u||_{(s_k)}||u||)
\end{aligned}
$$

Proof: estimate on $i[Q, Q']$ with $Q \in \mathcal{Q}_k$ and $Q' \in \mathcal{Q}_1$

$$\begin{aligned}
\|[Q_1, Q_k]u\|^2_{(s_{k+1})} &= (\langle D\rangle^{s_{k+1}}[Q_1, Q_k]u, \langle D\rangle^{s_{k+1}}[Q_1, Q_k]u) \\
&= ([Q_1, Q_k]u, \underbrace{\langle D\rangle^{2s_{k+1}}[Q_1, Q_k]}_{=:A\in OpS^{1+2s_{k+1}} = Op(S^{s_k})} u) \\
&= (Q_k u, Q_1^* Au) - (Q_1 u, Q_k^* Au) \\
&= (Q_k u, AQ_1 u) + (Q_k u, \underbrace{((Q_1^* - Q_1)A + [Q_1, A])}_{B\in Op(S^{s_k})} u) \\
&- (Q_1 u, AQ_k u) - (Q_1 u, \underbrace{((Q_k^* - Q_k)A + [Q_k, A])}_{C\in Op(S^{s_k})\subset Op(S^0)} u) \\
&= (A^* Q_k u, Q_1 u) + (B^* Q_k u, u) \\
&\quad - (Q_1 u, AQ_k u) - (Q_1 u, Cu) \\
&= O(\|Q_1 u\|_{(s_1)}\|Q_k u\|_{(s_k)} + \|Q_1 u\|_{(s_1)}\|u\| \\
&\quad + \|Q_k u\|_{(s_k)}\|u\|)
\end{aligned}$$

Proof: estimate on $i[Q, Q']$ with $Q \in \mathcal{Q}_k$ and $Q' \in \mathcal{Q}_1$

$$
\begin{aligned}
\left|\left|[Q_1, Q_k]u\right|\right|^2_{(s_{k+1})} &= (\langle D\rangle^{s_{k+1}}[Q_1, Q_k]u, \langle D\rangle^{s_{k+1}}[Q_1, Q_k]u) \\
&= \Big([Q_1, Q_k]u, \underbrace{\langle D\rangle^{2s_{k+1}}[Q_1, Q_k]}_{=:A\in OpS^{1+2s_{k+1}}=Op(S^{s_k})} u\Big) \\
&= (Q_k u, Q_1^* Au) - (Q_1 u, Q_k^* Au) \\
&= (Q_k u, AQ_1 u) + \Big(Q_k u, \underbrace{((Q_1^* - Q_1)A + [Q_1, A])}_{B\in Op(S^{s_k})} u\Big) \\
&- (Q_1 u, AQ_k u) - \Big(Q_1 u, \underbrace{((Q_k^* - Q_k)A + [Q_k, A])}_{C\in Op(S^{s_k})\subset Op(S^0)} u\Big) \\
&= (A^* Q_k u, Q_1 u) + (B^* Q_k u, u) \\
&\quad - (Q_1 u, AQ_k u) - (Q_1 u, Cu) \\
&= O(||Q_1 u||_{(s_1)}||Q_k u||_{(s_k)} + ||Q_1 u||_{(s_1)}||u|| \\
&\quad + ||Q_k u||_{(s_k)}||u||)
\end{aligned}
$$

Proof: estimate on $i[Q, Q']$ with $Q \in \mathcal{Q}_k$ and $Q' \in \mathcal{Q}_1$

$$
\begin{aligned}
||[Q_1, Q_k]u||^2_{(s_{k+1})} &= (\langle D\rangle^{s_{k+1}}[Q_1, Q_k]u, \langle D\rangle^{s_{k+1}}[Q_1, Q_k]u) \\
&= \big([Q_1, Q_k]u, \underbrace{\langle D\rangle^{2s_{k+1}}[Q_1, Q_k]}_{=:A\in OpS^{1+2s_{k+1}}=Op(S^{s_k})} u\big) \\
&= (Q_k u, Q_1^* Au) - (Q_1 u, Q_k^* Au) \\
&= (Q_k u, AQ_1 u) + \big(Q_k u, \underbrace{((Q_1^* - Q_1)A + [Q_1, A])}_{B\in Op(S^{s_k})} u\big) \\
&- (Q_1 u, AQ_k u) - \big(Q_1 u, \underbrace{((Q_k^* - Q_k)A + [Q_k, A])}_{C\in Op(S^{s_k})\subset Op(S^0)} u\big) \\
&= (A^* Q_k u, Q_1 u) + (B^* Q_k u, u) \\
&\quad - (Q_1 u, AQ_k u) - (Q_1 u, Cu) \\
&= O\big(||Q_1 u||_{(s_1)}||Q_k u||_{(s_k)} + ||Q_1 u||_{(s_1)}||u|| \\
&\quad + ||Q_k u||_{(s_k)}||u||\big)
\end{aligned}
$$

Proof: estimate on $i[Q, Q']$ with $Q \in \mathcal{Q}_k$ and $Q' \in \mathcal{Q}_1$

$$
\begin{aligned}
\left|\left|[Q_1, Q_k]u\right|\right|^2_{(s_{k+1})} &= (\langle D\rangle^{s_{k+1}}[Q_1, Q_k]u, \langle D\rangle^{s_{k+1}}[Q_1, Q_k]u) \\
&= \Big([Q_1, Q_k]u, \underbrace{\langle D\rangle^{2s_{k+1}}[Q_1, Q_k]}_{=:A\in OpS^{1+2s_{k+1}} = Op(S^{s_k})} u\Big) \\
&= (Q_k u, Q_1^* A u) - (Q_1 u, Q_k^* A u) \\
&= (Q_k u, A Q_1 u) + \Big(Q_k u, \underbrace{((Q_1^* - Q_1)A + [Q_1, A])}_{B\in Op(S^{s_k})} u\Big) \\
&- (Q_1 u, A Q_k u) - \Big(Q_1 u, \underbrace{((Q_k^* - Q_k)A + [Q_k, A])}_{C\in Op(S^{s_k})\subset Op(S^0)} u\Big) \\
&= (A^* Q_k u, Q_1 u) + (B^* Q_k u, u) \\
&\quad - (Q_1 u, A Q_k u) - (Q_1 u, C u) \\
&= O(||Q_1 u||_{(s_1)} ||Q_k u||_{(s_k)} + ||Q_1 u||_{(s_1)} ||u|| \\
&\quad + ||Q_k u||_{(s_k)} ||u||)
\end{aligned}
$$

Proof: estimate on $i[Q, Q']$ with $Q \in \mathcal{Q}_k$ and $Q' \in \mathcal{Q}_1$

$$
\begin{aligned}
\left|\left|[Q_1, Q_k]u\right|\right|^2_{(s_{k+1})} &= (\langle D\rangle^{s_{k+1}}[Q_1, Q_k]u, \langle D\rangle^{s_{k+1}}[Q_1, Q_k]u) \\
&= ([Q_1, Q_k]u, \underbrace{\langle D\rangle^{2s_{k+1}}[Q_1, Q_k]}_{=:A \in OpS^{1+2s_{k+1}} = Op(S^{s_k})} u) \\
&= (Q_k u, Q_1^* A u) - (Q_1 u, Q_k^* A u) \\
&= (Q_k u, A Q_1 u) + (Q_k u, \underbrace{((Q_1^* - Q_1)A + [Q_1, A])}_{B \in Op(S^{s_k})} u) \\
&- (Q_1 u, A Q_k u) - (Q_1 u, \underbrace{((Q_k^* - Q_k)A + [Q_k, A])}_{C \in Op(S^{s_k}) \subset Op(S^0)} u) \\
&= (A^* Q_k u, Q_1 u) + (B^* Q_k u, u) \\
&\quad - (Q_1 u, A Q_k u) - (Q_1 u, C u) \\
&= O(||Q_1 u||_{(s_1)} ||Q_k u||_{(s_k)} + ||Q_1 u||_{(s_1)} ||u|| \\
&\quad + ||Q_k u||_{(s_k)} ||u||)
\end{aligned}
$$

Proof: estimate on $i[Q, Q']$ with $Q \in \mathcal{Q}_k$ and $Q' \in \mathcal{Q}_1$

$$
\begin{aligned}
\|[Q_1, Q_k]u\|^2_{(s_{k+1})} &= (\langle D\rangle^{s_{k+1}}[Q_1, Q_k]u, \langle D\rangle^{s_{k+1}}[Q_1, Q_k]u) \\
&= \big([Q_1, Q_k]u, \underbrace{\langle D\rangle^{2s_{k+1}}[Q_1, Q_k]}_{=:A\in OpS^{1+2s_{k+1}}=Op(S^{s_k})} u\big) \\
&= (Q_k u, Q_1^* Au) - (Q_1 u, Q_k^* Au) \\
&= (Q_k u, AQ_1 u) + \big(Q_k u, \underbrace{((Q_1^* - Q_1)A + [Q_1, A])}_{B\in Op(S^{s_k})} u\big) \\
&- (Q_1 u, AQ_k u) - \big(Q_1 u, \underbrace{((Q_k^* - Q_k)A + [Q_k, A])}_{C\in Op(S^{s_k})\subset Op(S^0)} u\big) \\
&= (A^* Q_k u, Q_1 u) + (B^* Q_k u, u) \\
&\quad - (Q_1 u, AQ_k u) - (Q_1 u, Cu) \\
&= O\big(\|Q_1 u\|_{(s_1)}\|Q_k u\|_{(s_k)} + \|Q_1 u\|_{(s_1)}\|u\| \\
&\quad + \|Q_k u\|_{(s_k)}\|u\|\big)
\end{aligned}
$$

Proof: estimate on $i[Q, Q']$ with $Q \in \mathcal{Q}_k$ and $Q' \in \mathcal{Q}_1$

$$
\begin{aligned}
\left|\left|[Q_1, Q_k]u\right|\right|^2_{(s_{k+1})} &= (\langle D\rangle^{s_{k+1}}[Q_1, Q_k]u, \langle D\rangle^{s_{k+1}}[Q_1, Q_k]u) \\
&= \Big([Q_1, Q_k]u, \underbrace{\langle D\rangle^{2s_{k+1}}[Q_1, Q_k]}_{=:A\in OpS^{1+2s_{k+1}}=Op(S^{s_k})} u\Big) \\
&= (Q_k u, Q_1^* A u) - (Q_1 u, Q_k^* A u) \\
&= (Q_k u, A Q_1 u) + \Big(Q_k u, \underbrace{((Q_1^* - Q_1)A + [Q_1, A])}_{B\in Op(S^{s_k})} u\Big) \\
&- (Q_1 u, A Q_k u) - \Big(Q_1 u, \underbrace{((Q_k^* - Q_k)A + [Q_k, A])}_{C\in Op(S^{s_k})\subset Op(S^0)} u\Big) \\
&= (A^* Q_k u, Q_1 u) + (B^* Q_k u, u) \\
&\quad - (Q_1 u, A Q_k u) - (Q_1 u, C u) \\
&= O\Big(||Q_1 u||_{(s_1)} ||Q_k u||_{(s_k)} + ||Q_1 u||_{(s_1)} ||u|| \\
&\quad + ||Q_k u||_{(s_k)} ||u||\Big)
\end{aligned}
$$

Proof: estimate on $i[Q, Q']$ with $Q \in \mathcal{Q}_{k-2}$ and $Q' = P''$

$$\begin{aligned} \|[P'', Q_{k-1}]u\|^2_{(s_{k+1})} &= ([P'', Q_{k-1}]u, Au) \qquad (\text{with } A \in Op(S^{s_k})) \\ &= i(Q_{k-1}u, (P' - P)Au) + i((P - P')u, Q^*_{k-1}Au) \end{aligned}$$

where

$$\begin{aligned} (Pu, Q^*_{k-1}Au) &= (Pu, AQ_{k-1}u) + O(\|Pu\|\,\|u\|) \\ &= O(\|Pu\|\,\|Q_{k-1}u\|_{(s_k)}) + O(\|Pu\|\,\|u\|) \\ &= O(\|Pu\|\,\|Q_{k-1}u\|_{(s_{k-1})}) + O(\|Pu\|\,\|u\|) \end{aligned}$$

$$\begin{aligned} |(Q_{k-1}u, PAu)| &\lesssim \|Q_{k-1}u\|_{(s_{k-1})}\|PAu\|_{(-s_{k-1})} \\ &\lesssim \|Q_{k-1}u\|_{(s_{k-1})}(\|APu\|_{(-s_{k-1})} + \|[P, A]u\|_{(-s_{k-1})}) \\ &\lesssim \|Q_{k-1}u\|_{(s_{k-1})}(\|Pu\|_{(s_k - s_{k-1} \leq 0)} + \|[P, A]u\|_{(-s_{k-1})} \end{aligned}$$

in which $[P, A] = -2\sum_1^r Op(S^{s_k})L_j + Op(S^{s_k})$ so

$$\|[P, A]u\|_{(-s_{k-1})} \lesssim \sum_1^r \|L_j u\|_{(s_k - s_{k-1} \leq 0)} + \|u\|_{(s_k - s_{k-1} \leq 0)}$$

Proof: estimate on $i[Q, Q']$ with $Q \in \mathcal{Q}_{k-2}$ and $Q' = P''$

$$\begin{aligned}
\left|\left|[P'', Q_{k-1}]u\right|\right|^2_{(s_{k+1})} &= ([P'', Q_{k-1}]u, Au) \qquad (\text{with } A \in Op(S^{s_k})) \\
&= i(Q_{k-1}u, (P' - P)Au) + i((P - P')u, Q^*_{k-1}Au)
\end{aligned}$$

where

$$\begin{aligned}
(Pu, Q^*_{k-1}Au) &= (Pu, AQ_{k-1}u) + O(||Pu||\,||u||) \\
&= O(||Pu||\,||Q_{k-1}u||_{(s_k)}) + O(||Pu||\,||u||) \\
&= O(||Pu||\,||Q_{k-1}u||_{(s_{k-1})}) + O(||Pu||\,||u||)
\end{aligned}$$

$$\begin{aligned}
|(Q_{k-1}u, PAu)| &\lesssim ||Q_{k-1}u||_{(s_{k-1})}||PAu||_{(-s_{k-1})} \\
&\lesssim ||Q_{k-1}u||_{(s_{k-1})}\left(||APu||_{(-s_{k-1})} + ||[P, A]u||_{(-s_{k-1})}\right) \\
&\lesssim ||Q_{k-1}u||_{(s_{k-1})}\left(||Pu||_{(s_k - s_{k-1} \leq 0)} + ||[P, A]u||_{(-s_{k-1})}\right)
\end{aligned}$$

in which $[P, A] = -2\sum_1^r Op(S^{s_k})L_j + Op(S^{s_k})$ so

$$||[P, A]u||_{(-s_{k-1})} \lesssim \sum_1^r ||L_j u||_{(s_k - s_{k-1} \leq 0)} + ||u||_{(s_k - s_{k-1} \leq 0)}$$

Proof: estimate on $i[Q, Q']$ with $Q \in \mathcal{Q}_{k-2}$ and $Q' = P''$

$$
\begin{aligned}
\left|\left|[P'', Q_{k-1}]u\right|\right|^2_{(s_{k+1})} &= \big([P'', Q_{k-1}]u, Au\big) \qquad \text{(with } A \in Op(S^{s_k})\text{)} \\
&= i(Q_{k-1}u, (P' - P)Au) + i((P - P')u, Q^*_{k-1}Au)
\end{aligned}
$$

where

$$
\begin{aligned}
(Pu, Q^*_{k-1}Au) &= (Pu, AQ_{k-1}u) + O(||Pu||\,||u||) \\
&= O(||Pu||\,||Q_{k-1}u||_{(s_k)}) + O(||Pu||\,||u||) \\
&= O(||Pu||\,||Q_{k-1}u||_{(s_{k-1})}) + O(||Pu||\,||u||)
\end{aligned}
$$

$$
\begin{aligned}
|(Q_{k-1}u, PAu)| &\lesssim ||Q_{k-1}u||_{(s_{k-1})}||PAu||_{(-s_{k-1})} \\
&\lesssim ||Q_{k-1}u||_{(s_{k-1})}\big(||APu||_{(-s_{k-1})} + ||[P, A]u||_{(-s_{k-1})}\big) \\
&\lesssim ||Q_{k-1}u||_{(s_{k-1})}\big(||Pu||_{(s_k - s_{k-1} \le 0)} + ||[P, A]u||_{(-s_{k-1})}
\end{aligned}
$$

in which $[P, A] = -2\sum_1^r Op(S^{s_k})L_j + Op(S^{s_k})$ so

$$
||[P, A]u||_{(-s_{k-1})} \lesssim \sum^r ||L_j u||_{(s_k - s_{k-1} \le 0)} + ||u||_{(s_k - s_{k-1} \le 0)}
$$

Proof: estimate on $i[Q, Q']$ with $Q \in \mathcal{Q}_{k-2}$ and $Q' = P''$

$$\begin{aligned} \big|\big|[P'', Q_{k-1}]u\big|\big|^2_{(s_{k+1})} &= \big([P'', Q_{k-1}]u, Au\big) \qquad (\text{with } A \in Op(S^{s_k})) \\ &= i(Q_{k-1}u, (P' - P)Au) + i((P - P')u, Q^*_{k-1}Au) \end{aligned}$$

where

$$\begin{aligned} (Pu, Q^*_{k-1}Au) &= (Pu, AQ_{k-1}u) + O(||Pu||\,||u||) \\ &= O(||Pu||\,||Q_{k-1}u||_{(s_k)}) + O(||Pu||\,||u||) \\ &= O(||Pu||\,||Q_{k-1}u||_{(s_{k-1})}) + O(||Pu||\,||u||) \end{aligned}$$

$$\begin{aligned} |(Q_{k-1}u, PAu)| &\lesssim ||Q_{k-1}u||_{(s_{k-1})}||PAu||_{(-s_{k-1})} \\ &\lesssim ||Q_{k-1}u||_{(s_{k-1})}\big(||APu||_{(-s_{k-1})} + ||[P, A]u||_{(-s_{k-1})}\big) \\ &\lesssim ||Q_{k-1}u||_{(s_{k-1})}\big(||Pu||_{(s_k - s_{k-1} \leq 0)} + ||[P, A]u||_{(-s_{k-1})} \end{aligned}$$

in which $[P, A] = -2\sum_1^r Op(S^{s_k})L_j + Op(S^{s_k})$ so

$$||[P, A]u||_{(-s_{k-1})} \lesssim \sum_1^r ||L_j u||_{(s_k - s_{k-1} \leq 0)} + ||u||_{(s_k - s_{k-1} \leq 0)}$$

Proof: estimate on $i[Q, Q']$ with $Q \in \mathcal{Q}_{k-2}$ and $Q' = P''$

$$\left|\left|[P'', Q_{k-1}]u\right|\right|^2_{(s_{k+1})} = \left([P'', Q_{k-1}]u, Au\right) \qquad (\text{with } A \in Op(S^{s_k}))$$
$$= i(Q_{k-1}u, (P' - P)Au) + i((P - P')u, Q^*_{k-1}Au)$$

where

$$(Pu, Q^*_{k-1}Au) = (Pu, AQ_{k-1}u) + O(||Pu||\,||u||)$$
$$= O(||Pu||\,||Q_{k-1}u||_{(s_k)}) + O(||Pu||\,||u||)$$
$$= O(||Pu||\,||Q_{k-1}u||_{(s_{k-1})}) + O(||Pu||\,||u||)$$

$$|(Q_{k-1}u, PAu)| \lesssim ||Q_{k-1}u||_{(s_{k-1})}||PAu||_{(-s_{k-1})}$$
$$\lesssim ||Q_{k-1}u||_{(s_{k-1})}\left(||APu||_{(-s_{k-1})} + ||[P, A]u||_{(-s_{k-1})}\right)$$
$$\lesssim ||Q_{k-1}u||_{(s_{k-1})}\left(||Pu||_{(s_k - s_{k-1} \leq 0)} + ||[P, A]u||_{(-s_{k-1})}\right)$$

in which $[P, A] = -2\sum_1^r Op(S^{s_k})L_j + Op(S^{s_k})$ so

$$||[P, A]u||_{(-s_{k-1})} \lesssim \sum_1^r ||L_j u||_{(s_k - s_{k-1} \leq 0)} + ||u||_{(s_k - s_{k-1} \leq 0)}$$

Proof: estimate on $i[Q, Q']$ with $Q \in \mathcal{Q}_{k-2}$ and $Q' = P''$

$$
\begin{aligned}
\big|\big|[P'', Q_{k-1}]u\big|\big|^2_{(s_{k+1})} &= \big([P'', Q_{k-1}]u, Au\big) \qquad (\text{with } A \in Op(S^{s_k}))\\
&= i(Q_{k-1}u, (P' - P)Au) + i((P - P')u, Q^*_{k-1}Au)
\end{aligned}
$$

where

$$
\begin{aligned}
(Pu, Q^*_{k-1}Au) &= (Pu, AQ_{k-1}u) + O(||Pu||\,||u||)\\
&= O(||Pu||\,||Q_{k-1}u||_{(s_k)}) + O(||Pu||\,||u||)\\
&= O(||Pu||\,||Q_{k-1}u||_{(s_{k-1})}) + O(||Pu||\,||u||)
\end{aligned}
$$

$$
\begin{aligned}
|(Q_{k-1}u, PAu)| &\lesssim ||Q_{k-1}u||_{(s_{k-1})}||PAu||_{(-s_{k-1})}\\
&\lesssim ||Q_{k-1}u||_{(s_{k-1})}\big(||APu||_{(-s_{k-1})} + ||[P, A]u||_{(-s_{k-1})}\big)\\
&\lesssim ||Q_{k-1}u||_{(s_{k-1})}\big(||Pu||_{(s_k - s_{k-1} \leq 0)} + ||[P, A]u||_{(-s_{k-1})}
\end{aligned}
$$

in which $[P, A] = -2\sum_1^r Op(S^{s_k})L_j + Op(S^{s_k})$ so

$$
||[P, A]u||_{(-s_{k-1})} \lesssim \sum^r ||L_j u||_{(s_k - s_{k-1} \leq 0)} + ||u||_{(s_k - s_{k-1} \leq 0)}
$$

Proof: estimate on $i[Q, Q']$ with $Q \in \mathcal{Q}_{k-2}$ and $Q' = P''$

$$
\begin{aligned}
\big|\big|[P'', Q_{k-1}]u\big|\big|^2_{(s_{k+1})} &= \big([P'', Q_{k-1}]u, Au\big) \qquad (\text{with } A \in Op(S^{s_k})) \\
&= i(Q_{k-1}u, (P' - P)Au) + i((P - P')u, Q^*_{k-1}Au)
\end{aligned}
$$

where

$$
\begin{aligned}
(Pu, Q^*_{k-1}Au) &= (Pu, AQ_{k-1}u) + O(||Pu||\,||u||) \\
&= O(||Pu||\,||Q_{k-1}u||_{(s_k)}) + O(||Pu||\,||u||) \\
&= O(||Pu||\,||Q_{k-1}u||_{(s_{k-1})}) + O(||Pu||\,||u||)
\end{aligned}
$$

$$
\begin{aligned}
|(Q_{k-1}u, PAu)| &\lesssim ||Q_{k-1}u||_{(s_{k-1})}||PAu||_{(-s_{k-1})} \\
&\lesssim ||Q_{k-1}u||_{(s_{k-1})}\big(||APu||_{(-s_{k-1})} + ||[P, A]u||_{(-s_{k-1})}\big) \\
&\lesssim ||Q_{k-1}u||_{(s_{k-1})}\big(||Pu||_{(s_k - s_{k-1} \leq 0)} + ||[P, A]u||_{(-s_{k-1})}
\end{aligned}
$$

in which $[P, A] = -2\sum_1^r Op(S^{s_k})L_j + Op(S^{s_k})$ so

$$
||[P, A]u||_{(-s_{k-1})} \lesssim \sum_1^r ||L_j u||_{(s_k - s_{k-1} \leq 0)} + ||u||_{(s_k - s_{k-1} \leq 0)}
$$

Proof: estimate on $i[Q, Q']$ with $Q \in \mathcal{Q}_{k-2}$ and $Q' = P''$

$$\left|\left|[P'', Q_{k-1}]u\right|\right|^2_{(s_{k+1})} = \left([P'', Q_{k-1}]u, Au\right) \qquad (\text{with } A \in Op(S^{s_k}))$$

$$= i(Q_{k-1}u, (P' - P)Au) + i((P - P')u, Q^*_{k-1}Au)$$

where

$$(Pu, Q^*_{k-1}Au) = (Pu, AQ_{k-1}u) + O(||Pu||\ ||u||)$$

$$= O(||Pu||\ ||Q_{k-1}u||_{(s_k)}) + O(||Pu||\ ||u||)$$

$$= O(||Pu||\ ||Q_{k-1}u||_{(s_{k-1})}) + O(||Pu||\ ||u||)$$

$$|(Q_{k-1}u, PAu)| \lesssim ||Q_{k-1}u||_{(s_{k-1})}||PAu||_{(-s_{k-1})}$$

$$\lesssim ||Q_{k-1}u||_{(s_{k-1})}\left(||APu||_{(-s_{k-1})} + ||[P, A]u||_{(-s_{k-1})}\right)$$

$$\lesssim ||Q_{k-1}u||_{(s_{k-1})}\left(||Pu||_{(s_k - s_{k-1} \leq 0)} + ||[P, A]u||_{(-s_{k-1})}\right)$$

in which $[P, A] = -2\sum_1^r Op(S^{s_k})L_j + Op(S^{s_k})$ so

$$||[P, A]u||_{(-s_{k-1})} \lesssim \sum^{r} ||L_j u||_{(s_k - s_{k-1} \leq 0)} + ||u||_{(s_k - s_{k-1} \leq 0)}$$

Proof: estimate on $i[Q, Q']$ with $Q \in \mathcal{Q}_{k-2}$ and $Q' = P''$

$$
\begin{aligned}
\left|\left|[P'', Q_{k-1}]u\right|\right|^2_{(s_{k+1})} &= \left([P'', Q_{k-1}]u, Au\right) \qquad (\text{with } A \in Op(S^{s_k})) \\
&= i(Q_{k-1}u, (P' - P)Au) + i((P - P')u, Q^*_{k-1}Au)
\end{aligned}
$$

where

$$
\begin{aligned}
(Pu, Q^*_{k-1}Au) &= (Pu, AQ_{k-1}u) + O(||Pu||\ ||u||) \\
&= O(||Pu||\ ||Q_{k-1}u||_{(s_k)}) + O(||Pu||\ ||u||) \\
&= O(||Pu||\ ||Q_{k-1}u||_{(s_{k-1})}) + O(||Pu||\ ||u||)
\end{aligned}
$$

$$
\begin{aligned}
|(Q_{k-1}u, PAu)| &\lesssim ||Q_{k-1}u||_{(s_{k-1})}||PAu||_{(-s_{k-1})} \\
&\lesssim ||Q_{k-1}u||_{(s_{k-1})}\left(||APu||_{(-s_{k-1})} + ||[P, A]u||_{(-s_{k-1})}\right) \\
&\lesssim ||Q_{k-1}u||_{(s_{k-1})}\left(||Pu||_{(s_k - s_{k-1} \leq 0)} + ||[P, A]u||_{(-s_{k-1})}\right)
\end{aligned}
$$

in which $[P, A] = -2\sum_1^r Op(S^{s_k})L_j + Op(S^{s_k})$ so

$$
||[P, A]u||_{(-s_{k-1})} \lesssim \sum_1^r ||L_j u||_{(s_k - s_{k-1} \leq 0)} + ||u||_{(s_k - s_{k-1} \leq 0)}
$$

Proof: estimate on $i[Q, Q']$ with $Q \in \mathcal{Q}_{k-2}$ and $Q' = P''$

$$
\begin{aligned}
\left|\left|[P'', Q_{k-1}]u\right|\right|^2_{(s_{k+1})} &= \left([P'', Q_{k-1}]u, Au\right) \qquad (\text{with } A \in Op(S^{s_k})) \\
&= i(Q_{k-1}u, (P' - P)Au) + i((P - P')u, Q^*_{k-1}Au)
\end{aligned}
$$

where

$$
\begin{aligned}
(Pu, Q^*_{k-1}Au) &= (Pu, AQ_{k-1}u) + O(||Pu||\ ||u||) \\
&= O(||Pu||\ ||Q_{k-1}u||_{(s_k)}) + O(||Pu||\ ||u||) \\
&= O(||Pu||\ ||Q_{k-1}u||_{(s_{k-1})}) + O(||Pu||\ ||u||)
\end{aligned}
$$

$$
\begin{aligned}
|(Q_{k-1}u, PAu)| &\lesssim ||Q_{k-1}u||_{(s_{k-1})}||PAu||_{(-s_{k-1})} \\
&\lesssim ||Q_{k-1}u||_{(s_{k-1})}\left(||APu||_{(-s_{k-1})} + ||[P, A]u||_{(-s_{k-1})}\right) \\
&\lesssim ||Q_{k-1}u||_{(s_{k-1})}\left(||Pu||_{(s_k - s_{k-1} \leq 0)} + ||[P, A]u||_{(-s_{k-1})}\right)
\end{aligned}
$$

in which $[P, A] = -2\sum_1^r Op(S^{s_k})L_j + Op(S^{s_k})$ so

$$
||[P, A]u||_{(-s_{k-1})} \lesssim \sum_1^r ||L_j u||_{(s_k - s_{k-1} \leq 0)} + ||u||_{(s_k - s_{k-1} \leq 0)}
$$

Proof: estimate on $i[Q, Q']$ with $Q \in \mathcal{Q}_{k-2}$ and $Q' = P''$

$$\begin{aligned} \left|\left|[P'', Q_{k-1}]u\right|\right|^2_{(s_{k+1})} &= \big([P'', Q_{k-1}]u, Au\big) \qquad (\text{with } A \in Op(S^{s_k})) \\ &= i(Q_{k-1}u, (P' - P)Au) + i((P - P')u, Q^*_{k-1}Au) \end{aligned}$$

where

$$\begin{aligned} (Pu, Q^*_{k-1}Au) &= (Pu, AQ_{k-1}u) + O(||Pu||\,||u||) \\ &= O(||Pu||\,||Q_{k-1}u||_{(s_k)}) + O(||Pu||\,||u||) \\ &= O(||Pu||\,||Q_{k-1}u||_{(s_{k-1})}) + O(||Pu||\,||u||) \end{aligned}$$

$$\begin{aligned} |(Q_{k-1}u, PAu)| &\lesssim ||Q_{k-1}u||_{(s_{k-1})}||PAu||_{(-s_{k-1})} \\ &\lesssim ||Q_{k-1}u||_{(s_{k-1})}\big(||APu||_{(-s_{k-1})} + ||[P, A]u||_{(-s_{k-1})}\big) \\ &\lesssim ||Q_{k-1}u||_{(s_{k-1})}\big(||Pu||_{(s_k - s_{k-1} \leq 0)} + ||[P, A]u||_{(-s_{k-1})} \end{aligned}$$

in which $[P, A] = -2\sum_1^r Op(S^{s_k})L_j + Op(S^{s_k})$ so

$$||[P, A]u||_{(-s_{k-1})} \lesssim \sum_1^r ||L_j u||_{(s_k - s_{k-1} \leq 0)} + ||u||_{(s_k - s_{k-1} \leq 0)}$$

Proof: estimate on $i[Q, Q']$ with $Q \in \mathcal{Q}_{k-2}$ and $Q' = P''$

$$\begin{aligned}
\left|\left|[P'', Q_{k-1}]u\right|\right|^2_{(s_{k+1})} &= \left([P'', Q_{k-1}]u, Au\right) \qquad (\text{with } A \in Op(S^{s_k})) \\
&= i(Q_{k-1}u, (P' - P)Au) + i((P - P')u, Q^*_{k-1}Au)
\end{aligned}$$

where

$$\begin{aligned}
(Pu, Q^*_{k-1}Au) &= (Pu, AQ_{k-1}u) + O(||Pu||\ ||u||) \\
&= O(||Pu||\ ||Q_{k-1}u||_{(s_k)}) + O(||Pu||\ ||u||) \\
&= O(||Pu||\ ||Q_{k-1}u||_{(s_{k-1})}) + O(||Pu||\ ||u||)
\end{aligned}$$

$$\begin{aligned}
|(Q_{k-1}u, PAu)| &\lesssim ||Q_{k-1}u||_{(s_{k-1})}||PAu||_{(-s_{k-1})} \\
&\lesssim ||Q_{k-1}u||_{(s_{k-1})}\left(||APu||_{(-s_{k-1})} + ||[P, A]u||_{(-s_{k-1})}\right) \\
&\lesssim ||Q_{k-1}u||_{(s_{k-1})}\left(||Pu||_{(s_k - s_{k-1} \leq 0)} + ||[P, A]u||_{(-s_{k-1})}\right)
\end{aligned}$$

in which $[P, A] = -2\sum_1^r Op(S^{s_k})L_j + Op(S^{s_k})$ so

$$||[P, A]u||_{(-s_{k-1})} \lesssim \sum_1^r ||L_j u||_{(s_k - s_{k-1} \leq 0)} + ||u||_{(s_k - s_{k-1} \leq 0)}$$

Proof: estimate on $i[Q, Q']$ with $Q \in \mathcal{Q}_{k-2}$ and $Q' = P''$

$$\begin{aligned} \big|\big|[P'', Q_{k-1}]u\big|\big|^2_{(s_{k+1})} &= \big([P'', Q_{k-1}]u, Au\big) \qquad (\text{with } A \in Op(S^{s_k})) \\ &= i(Q_{k-1}u, (P' - P)Au) + i((P - P')u, Q^*_{k-1}Au) \end{aligned}$$

where

$$\begin{aligned} (Pu, Q^*_{k-1}Au) &= (Pu, AQ_{k-1}u) + O(||Pu||\,||u||) \\ &= O(||Pu||\,||Q_{k-1}u||_{(s_k)}) + O(||Pu||\,||u||) \\ &= O(||Pu||\,||Q_{k-1}u||_{(s_{k-1})}) + O(||Pu||\,||u||) \end{aligned}$$

$$\begin{aligned} |(Q_{k-1}u, PAu)| &\lesssim ||Q_{k-1}u||_{(s_{k-1})}||PAu||_{(-s_{k-1})} \\ &\lesssim ||Q_{k-1}u||_{(s_{k-1})}\big(||APu||_{(-s_{k-1})} + ||[P, A]u||_{(-s_{k-1})}\big) \\ &\lesssim ||Q_{k-1}u||_{(s_{k-1})}\big(||Pu||_{(s_k - s_{k-1} \leq 0)} + ||[P, A]u||_{(-s_{k-1})} \end{aligned}$$

in which $[P, A] = -2\sum_1^r Op(S^{s_k})L_j + Op(S^{s_k})$ so

$$||[P, A]u||_{(-s_{k-1})} \lesssim \sum^r ||L_j u||_{(s_k - s_{k-1} \leq 0)} + ||u||_{(s_k - s_{k-1} \leq 0)}$$

Proof: estimate on $i[Q, Q']$ with $Q \in \mathcal{Q}_{k-2}$ and $Q' = P''$

$$
\begin{aligned}
\left|\left|[P'', Q_{k-1}]u\right|\right|^2_{(s_{k+1})} &= \left([P'', Q_{k-1}]u, Au\right) \qquad (\text{with } A \in Op(S^{s_k})) \\
&= i(Q_{k-1}u, (P' - P)Au) + i((P - P')u, Q^*_{k-1}Au)
\end{aligned}
$$

where

$$
\begin{aligned}
(Pu, Q^*_{k-1}Au) &= (Pu, AQ_{k-1}u) + O(||Pu||\ ||u||) \\
&= O(||Pu||\ ||Q_{k-1}u||_{(s_k)}) + O(||Pu||\ ||u||) \\
&= O(||Pu||\ ||Q_{k-1}u||_{(s_{k-1})}) + O(||Pu||\ ||u||)
\end{aligned}
$$

$$
\begin{aligned}
|(Q_{k-1}u, PAu)| &\lesssim ||Q_{k-1}u||_{(s_{k-1})}||PAu||_{(-s_{k-1})} \\
&\lesssim ||Q_{k-1}u||_{(s_{k-1})}\left(||APu||_{(-s_{k-1})} + ||[P, A]u||_{(-s_{k-1})}\right) \\
&\lesssim ||Q_{k-1}u||_{(s_{k-1})}\left(||Pu||_{(s_k - s_{k-1} \leq 0)} + ||[P, A]u||_{(-s_{k-1})}\right)
\end{aligned}
$$

in which $[P, A] = -2\sum_1^r Op(S^{s_k})L_j + Op(S^{s_k})$ so

$$
||[P, A]u||_{(-s_{k-1})} \lesssim \sum^r ||L_j u||_{(s_k - s_{k-1} \leq 0)} + ||u||_{(s_k - s_{k-1} \leq 0)}
$$

Proof: estimate on $i[Q, Q']$ with $Q \in \mathcal{Q}_{k-2}$ and $Q' = P''$

$$
\begin{aligned}
\left|\left|[P'', Q_{k-1}]u\right|\right|^2_{(s_{k+1})} &= \left([P'', Q_{k-1}]u, Au\right) \qquad (\text{with } A \in Op(S^{s_k})) \\
&= i(Q_{k-1}u, (P' - P)Au) + i((P - P')u, Q^*_{k-1}Au)
\end{aligned}
$$

where

$$
\begin{aligned}
(Pu, Q^*_{k-1}Au) &= (Pu, AQ_{k-1}u) + O(||Pu||\ ||u||) \\
&= O(||Pu||\ ||Q_{k-1}u||_{(s_k)}) + O(||Pu||\ ||u||) \\
&= O(||Pu||\ ||Q_{k-1}u||_{(s_{k-1})}) + O(||Pu||\ ||u||)
\end{aligned}
$$

$$
\begin{aligned}
|(Q_{k-1}u, PAu)| &\lesssim ||Q_{k-1}u||_{(s_{k-1})}||PAu||_{(-s_{k-1})} \\
&\lesssim ||Q_{k-1}u||_{(s_{k-1})}\left(||APu||_{(-s_{k-1})} + ||[P, A]u||_{(-s_{k-1})}\right) \\
&\lesssim ||Q_{k-1}u||_{(s_{k-1})}\left(||Pu||_{(s_k - s_{k-1} \leq 0)} + ||[P, A]u||_{(-s_{k-1})}\right)
\end{aligned}
$$

in which $[P, A] = -2\sum_1^r Op(S^{s_k})L_j + Op(S^{s_k})$ so

$$
||[P, A]u||_{(-s_{k-1})} \lesssim \sum_1^r ||L_j u||_{(s_k - s_{k-1} \leq 0)} + ||u||_{(s_k - s_{k-1} \leq 0)}
$$

Proof: estimate on $i[Q, Q']$ with $Q \in \mathcal{Q}_{k-2}$ and $Q' = P''$

$$
\begin{aligned}
\|[P'', Q_{k-1}]u\|^2_{(s_{k+1})} &= ([P'', Q_{k-1}]u, Au) \qquad (\text{with } A \in Op(S^{s_k})) \\
&= i(Q_{k-1}u, (P' - P)Au) + i((P - P')u, Q^*_{k-1}Au)
\end{aligned}
$$

where

$$
\begin{aligned}
(Pu, Q^*_{k-1}Au) &= (Pu, AQ_{k-1}u) + O(\|Pu\|\,\|u\|) \\
&= O(\|Pu\|\,\|Q_{k-1}u\|_{(s_k)}) + O(\|Pu\|\,\|u\|) \\
&= O(\|Pu\|\,\|Q_{k-1}u\|_{(s_{k-1})}) + O(\|Pu\|\,\|u\|)
\end{aligned}
$$

$$
\begin{aligned}
|(Q_{k-1}u, PAu)| &\lesssim \|Q_{k-1}u\|_{(s_{k-1})}\|PAu\|_{(-s_{k-1})} \\
&\lesssim \|Q_{k-1}u\|_{(s_{k-1})}(\|APu\|_{(-s_{k-1})} + \|[P, A]u\|_{(-s_{k-1})}) \\
&\lesssim \|Q_{k-1}u\|_{(s_{k-1})}(\|Pu\|_{(s_k - s_{k-1} \leq 0)} + \|[P, A]u\|_{(-s_{k-1})}
\end{aligned}
$$

in which $[P, A] = -2\sum_1^r Op(S^{s_k})L_j + Op(S^{s_k})$ so

$$
\|[P, A]u\|_{(-s_{k-1})} \lesssim \sum^{r} \|L_j u\|_{(s_k - s_{k-1} \leq 0)} + \|u\|_{(s_k - s_{k-1} \leq 0)}
$$

Proof: estimate on $i[Q, Q']$ with $Q \in \mathcal{Q}_{k-2}$ and $Q' = P''$

$$
\begin{aligned}
\left|\left|[P'', Q_{k-1}]u\right|\right|^2_{(s_{k+1})} &= \big([P'', Q_{k-1}]u, Au\big) \qquad (\text{with } A \in Op(S^{s_k})) \\
&= i(Q_{k-1}u, (P' - P)Au) + i((P - P')u, Q^*_{k-1}Au)
\end{aligned}
$$

where

$$
\begin{aligned}
(Pu, Q^*_{k-1}Au) &= (Pu, AQ_{k-1}u) + O(||Pu||\ ||u||) \\
&= O(||Pu||\ ||Q_{k-1}u||_{(s_k)}) + O(||Pu||\ ||u||) \\
&= O(||Pu||\ ||Q_{k-1}u||_{(s_{k-1})}) + O(||Pu||\ ||u||)
\end{aligned}
$$

$$
\begin{aligned}
|(Q_{k-1}u, PAu)| &\lesssim ||Q_{k-1}u||_{(s_{k-1})}||PAu||_{(-s_{k-1})} \\
&\lesssim ||Q_{k-1}u||_{(s_{k-1})}\big(||APu||_{(-s_{k-1})} + ||[P, A]u||_{(-s_{k-1})}\big) \\
&\lesssim ||Q_{k-1}u||_{(s_{k-1})}\big(||Pu||_{(s_k - s_{k-1} \leq 0)} + ||[P, A]u||_{(-s_{k-1})}
\end{aligned}
$$

in which $[P, A] = -2\sum_1^r Op(S^{s_k})L_j + Op(S^{s_k})$ so

$$
||[P, A]u||_{(-s_{k-1})} \lesssim \sum^r ||L_j u||_{(s_k - s_{k-1} \leq 0)} + ||u||_{(s_k - s_{k-1} \leq 0)}
$$

Proof: estimate on $i[Q, Q']$ with $Q \in \mathcal{Q}_{k-2}$ and $Q' = P''$

$$
\begin{aligned}
\big|\big|[P'', Q_{k-1}]u\big|\big|^2_{(s_{k+1})} &= \big([P'', Q_{k-1}]u, Au\big) \qquad (\text{with } A \in Op(S^{s_k})) \\
&= i(Q_{k-1}u, (P' - P)Au) + i((P - P')u, Q^*_{k-1}Au)
\end{aligned}
$$

where

$$
\begin{aligned}
(Pu, Q^*_{k-1}Au) &= (Pu, AQ_{k-1}u) + O(||Pu||\ ||u||) \\
&= O(||Pu||\ ||Q_{k-1}u||_{(s_k)}) + O(||Pu||\ ||u||) \\
&= O(||Pu||\ ||Q_{k-1}u||_{(s_{k-1})}) + O(||Pu||\ ||u||)
\end{aligned}
$$

$$
\begin{aligned}
|(Q_{k-1}u, PAu)| &\lesssim ||Q_{k-1}u||_{(s_{k-1})}||PAu||_{(-s_{k-1})} \\
&\lesssim ||Q_{k-1}u||_{(s_{k-1})}\big(||APu||_{(-s_{k-1})} + ||[P, A]u||_{(-s_{k-1})}\big) \\
&\lesssim ||Q_{k-1}u||_{(s_{k-1})}\big(||Pu||_{(s_k - s_{k-1} \leq 0)} + ||[P, A]u||_{(-s_{k-1})}
\end{aligned}
$$

in which $[P, A] = -2\sum_1^r Op(S^{s_k})L_j + Op(S^{s_k})$ so

$$
||[P, A]u||_{(-s_{k-1})} \lesssim \sum_1^r ||L_j u||_{(s_k - s_{k-1} \leq 0)} + ||u||_{(s_k - s_{k-1} \leq 0)}
$$

Proof: estimate on $i[Q, Q']$ with $Q \in \mathcal{Q}_{k-2}$ and $Q' = P''$

$$
\begin{aligned}
||[P'', Q_{k-1}]u||^2_{(s_{k+1})} &= ([P'', Q_{k-1}]u, Au) \qquad (\text{with } A \in Op(S^{s_k})) \\
&= i(Q_{k-1}u, (P' - P)Au) + i((P - P')u, Q^*_{k-1}Au)
\end{aligned}
$$

where

$$
\begin{aligned}
(Pu, Q^*_{k-1}Au) &= (Pu, AQ_{k-1}u) + O(||Pu||\,||u||) \\
&= O(||Pu||\,||Q_{k-1}u||_{(s_k)}) + O(||Pu||\,||u||) \\
&= O(||Pu||\,||Q_{k-1}u||_{(s_{k-1})}) + O(||Pu||\,||u||)
\end{aligned}
$$

$$
\begin{aligned}
|(Q_{k-1}u, PAu)| &\lesssim ||Q_{k-1}u||_{(s_{k-1})}||PAu||_{(-s_{k-1})} \\
&\lesssim ||Q_{k-1}u||_{(s_{k-1})}\big(||APu||_{(-s_{k-1})} + ||[P, A]u||_{(-s_{k-1})}\big) \\
&\lesssim ||Q_{k-1}u||_{(s_{k-1})}\big(||Pu||_{(s_k - s_{k-1} \leq 0)} + ||[P, A]u||_{(-s_{k-1})}\big)
\end{aligned}
$$

in which $[P, A] = -2\sum_1^r Op(S^{s_k})L_j + Op(S^{s_k})$ so

$$
||[P, A]u||_{(-s_{k-1})} \lesssim \sum_1^r ||L_j u||_{(s_k - s_{k-1} \leq 0)} + ||u||_{(s_k - s_{k-1} \leq 0)}
$$

Proof: estimate on $i[Q, Q']$ with $Q \in \mathcal{Q}_{k-2}$ and $Q' = P''$

$$
\begin{aligned}
\big|\big|[P'', Q_{k-1}]u\big|\big|^2_{(s_{k+1})} &= \big([P'', Q_{k-1}]u, Au\big) \qquad (\text{with } A \in Op(S^{s_k})) \\
&= i(Q_{k-1}u, (P' - P)Au) + i((P - P')u, Q^*_{k-1}Au)
\end{aligned}
$$

where

$$
\begin{aligned}
(Pu, Q^*_{k-1}Au) &= (Pu, AQ_{k-1}u) + O(||Pu||\;||u||) \\
&= O(||Pu||\;||Q_{k-1}u||_{(s_k)}) + O(||Pu||\;||u||) \\
&= O(||Pu||\;||Q_{k-1}u||_{(s_{k-1})}) + O(||Pu||\;||u||)
\end{aligned}
$$

$$
\begin{aligned}
|(Q_{k-1}u, PAu)| &\lesssim ||Q_{k-1}u||_{(s_{k-1})}||PAu||_{(-s_{k-1})} \\
&\lesssim ||Q_{k-1}u||_{(s_{k-1})}\big(||APu||_{(-s_{k-1})} + ||[P, A]u||_{(-s_{k-1})}\big) \\
&\lesssim ||Q_{k-1}u||_{(s_{k-1})}\big(||Pu||_{(s_k - s_{k-1} \leq 0)} + ||[P, A]u||_{(-s_{k-1})}
\end{aligned}
$$

in which $[P, A] = -2\sum_1^r Op(S^{s_k})L_j + Op(S^{s_k})$ so

$$
||[P, A]u||_{(-s_{k-1})} \lesssim \sum^r ||L_j u||_{(s_k - s_{k-1} \leq 0)} + ||u||_{(s_k - s_{k-1} \leq 0)}
$$

Proof: estimate on $i[Q, Q']$ with $Q \in \mathcal{Q}_{k-2}$ and $Q' = P''$

$$
\begin{aligned}
||[P'', Q_{k-1}]u||^2_{(s_{k+1})} &= ([P'', Q_{k-1}]u, Au) \qquad (\text{with } A \in Op(S^{s_k})) \\
&= i(Q_{k-1}u, (P' - P)Au) + i((P - P')u, Q^*_{k-1}Au)
\end{aligned}
$$

where

$$
\begin{aligned}
(Pu, Q^*_{k-1}Au) &= (Pu, AQ_{k-1}u) + O(||Pu||\ ||u||) \\
&= O(||Pu||\ ||Q_{k-1}u||_{(s_k)}) + O(||Pu||\ ||u||) \\
&= O(||Pu||\ ||Q_{k-1}u||_{(s_{k-1})}) + O(||Pu||\ ||u||)
\end{aligned}
$$

$$
\begin{aligned}
|(Q_{k-1}u, PAu)| &\lesssim ||Q_{k-1}u||_{(s_{k-1})}||PAu||_{(-s_{k-1})} \\
&\lesssim ||Q_{k-1}u||_{(s_{k-1})}\big(||APu||_{(-s_{k-1})} + ||[P, A]u||_{(-s_{k-1})}\big) \\
&\lesssim ||Q_{k-1}u||_{(s_{k-1})}\big(||Pu||_{(s_k - s_{k-1} \leq 0)} + ||[P, A]u||_{(-s_{k-1})}
\end{aligned}
$$

in which $[P, A] = -2\sum_1^r Op(S^{s_k})L_j + Op(S^{s_k})$ so

$$
||[P, A]u||_{(-s_{k-1})} \lesssim \sum^r ||L_j u||_{(s_k - s_{k-1} \leq 0)} + ||u||_{(s_k - s_{k-1} \leq 0)}
$$

Proof: estimate on $i[Q, Q']$ with $Q \in \mathcal{Q}_{k-2}$ and $Q' = P''$

$$
\begin{aligned}
\left|\left|[P'', Q_{k-1}]u\right|\right|^2_{(s_{k+1})} &= \big([P'', Q_{k-1}]u, Au\big) \qquad (\text{with } A \in Op(S^{s_k})) \\
&= i(Q_{k-1}u, (P' - P)Au) + i((P - P')u, Q^*_{k-1}Au)
\end{aligned}
$$

where

$$
\begin{aligned}
(Pu, Q^*_{k-1}Au) &= (Pu, AQ_{k-1}u) + O(||Pu||\,||u||) \\
&= O(||Pu||\,||Q_{k-1}u||_{(s_k)}) + O(||Pu||\,||u||) \\
&= O(||Pu||\,||Q_{k-1}u||_{(s_{k-1})}) + O(||Pu||\,||u||)
\end{aligned}
$$

$$
\begin{aligned}
|(Q_{k-1}u, PAu)| &\lesssim ||Q_{k-1}u||_{(s_{k-1})}||PAu||_{(-s_{k-1})} \\
&\lesssim ||Q_{k-1}u||_{(s_{k-1})}\big(||APu||_{(-s_{k-1})} + ||[P, A]u||_{(-s_{k-1})}\big) \\
&\lesssim ||Q_{k-1}u||_{(s_{k-1})}\big(||Pu||_{(s_k - s_{k-1} \leq 0)} + ||[P, A]u||_{(-s_{k-1})}
\end{aligned}
$$

in which $[P, A] = -2\sum_1^r Op(S^{s_k})L_j + Op(S^{s_k})$ so

$$
||[P, A]u||_{(-s_{k-1})} \lesssim \sum^r ||L_j u||_{(s_k - s_{k-1} \leq 0)} + ||u||_{(s_k - s_{k-1} \leq 0)}
$$

Proof: estimate on $i[Q, Q']$ with $Q \in \mathcal{Q}_{k-2}$ and $Q' = P''$

$$
\begin{aligned}
\big|\big|[P'', Q_{k-1}]u\big|\big|^2_{(s_{k+1})} &= \big([P'', Q_{k-1}]u, Au\big) \qquad (\text{with } A \in Op(S^{s_k})) \\
&= i(Q_{k-1}u, (P' - P)Au) + i((P - P')u, Q^*_{k-1}Au)
\end{aligned}
$$

where

$$
\begin{aligned}
(Pu, Q^*_{k-1}Au) &= (Pu, AQ_{k-1}u) + O(||Pu||\ ||u||) \\
&= O(||Pu||\ ||Q_{k-1}u||_{(s_k)}) + O(||Pu||\ ||u||) \\
&= O(||Pu||\ ||Q_{k-1}u||_{(s_{k-1})}) + O(||Pu||\ ||u||)
\end{aligned}
$$

$$
\begin{aligned}
|(Q_{k-1}u, PAu)| &\lesssim ||Q_{k-1}u||_{(s_{k-1})}||PAu||_{(-s_{k-1})} \\
&\lesssim ||Q_{k-1}u||_{(s_{k-1})}\big(||APu||_{(-s_{k-1})} + ||[P, A]u||_{(-s_{k-1})}\big) \\
&\lesssim ||Q_{k-1}u||_{(s_{k-1})}\big(||Pu||_{(s_k - s_{k-1} \leq 0)} + ||[P, A]u||_{(-s_{k-1})}
\end{aligned}
$$

in which $[P, A] = -2\sum_1^r Op(S^{s_k})L_j + Op(S^{s_k})$ so

$$
||[P, A]u||_{(-s_{k-1})} \lesssim \sum^r ||L_j u||_{(s_k - s_{k-1} \leq 0)} + ||u||_{(s_k - s_{k-1} \leq 0)}
$$

Proof: estimate on $i[Q, Q']$ with $Q \in \mathcal{Q}_{k-2}$ and $Q' = P''$

$$\begin{aligned} \big|\big|[P'', Q_{k-1}]u\big|\big|^2_{(s_{k+1})} &= \big([P'', Q_{k-1}]u, Au\big) \qquad (\text{with } A \in Op(S^{s_k})) \\ &= i(Q_{k-1}u, (P' - P)Au) + i((P - P')u, Q^*_{k-1}Au) \end{aligned}$$

where

$$\begin{aligned} (Pu, Q^*_{k-1}Au) &= (Pu, AQ_{k-1}u) + O(||Pu||\ ||u||) \\ &= O(||Pu||\ ||Q_{k-1}u||_{(s_k)}) + O(||Pu||\ ||u||) \\ &= O(||Pu||\ ||Q_{k-1}u||_{(s_{k-1})}) + O(||Pu||\ ||u||) \end{aligned}$$

$$\begin{aligned} |(Q_{k-1}u, PAu)| &\lesssim ||Q_{k-1}u||_{(s_{k-1})}||PAu||_{(-s_{k-1})} \\ &\lesssim ||Q_{k-1}u||_{(s_{k-1})}\big(||APu||_{(-s_{k-1})} + ||[P, A]u||_{(-s_{k-1})}\big) \\ &\lesssim ||Q_{k-1}u||_{(s_{k-1})}\big(||Pu||_{(s_k - s_{k-1} \le 0)} + ||[P, A]u||_{(-s_{k-1})}\big) \end{aligned}$$

in which $[P, A] = -2\sum_1^r Op(S^{s_k})L_j + Op(S^{s_k})$ so

$$||[P, A]u||_{(-s_{k-1})} \lesssim \sum^{r} ||L_j u||_{(s_k - s_{k-1} \le 0)} + ||u||_{(s_k - s_{k-1} \le 0)}$$

Proof: estimate on $i[Q, Q']$ with $Q \in \mathcal{Q}_{k-2}$ and $Q' = P''$

$$
\begin{aligned}
||[P'', Q_{k-1}]u||^2_{(s_{k+1})} &= ([P'', Q_{k-1}]u, Au) \qquad (\text{with } A \in Op(S^{s_k})) \\
&= i(Q_{k-1}u, (P' - P)Au) + i((P - P')u, Q^*_{k-1}Au)
\end{aligned}
$$

where

$$
\begin{aligned}
(Pu, Q^*_{k-1}Au) &= (Pu, AQ_{k-1}u) + O(||Pu||\,||u||) \\
&= O(||Pu||\,||Q_{k-1}u||_{(s_k)}) + O(||Pu||\,||u||) \\
&= O(||Pu||\,||Q_{k-1}u||_{(s_{k-1})}) + O(||Pu||\,||u||)
\end{aligned}
$$

$$
\begin{aligned}
|(Q_{k-1}u, PAu)| &\lesssim ||Q_{k-1}u||_{(s_{k-1})}||PAu||_{(-s_{k-1})} \\
&\lesssim ||Q_{k-1}u||_{(s_{k-1})}\big(||APu||_{(-s_{k-1})} + ||[P, A]u||_{(-s_{k-1})}\big) \\
&\lesssim ||Q_{k-1}u||_{(s_{k-1})}\big(||Pu||_{(s_k - s_{k-1} \leq 0)} + ||[P, A]u||_{(-s_{k-1})}
\end{aligned}
$$

in which $[P, A] = -2\sum_1^r Op(S^{s_k})L_j + Op(S^{s_k})$ so

$$
||[P, A]u||_{(-s_{k-1})} \lesssim \sum_1^r ||L_ju||_{(s_k - s_{k-1} \leq 0)} + ||u||_{(s_k - s_{k-1} \leq 0)}
$$

Proof: estimate on $i[Q, Q']$ with $Q \in \mathcal{Q}_{k-2}$ and $Q' = P''$

It remains to estimate $(Q_{k-1}u, P'Au)$ and $(P'u, Q^*_{k-1}Au)$. First

$$|(P'u, Q^*_{k-1}Au)| \lesssim (\mathrm{Re}(Pu, u))^{\frac{1}{2}} (\mathrm{Re}(PQ^*_{k-1}Au, Q^*_{k-1}Au))^{\frac{1}{2}}$$

using again the P'-Cauchy-Schwarz inequality. Then, use

$$PQ^*_{k-1}A = \underbrace{Q^*_{k-1}A}_{\in Op(S^{1+s_k})} P + \sum_1^r Op(S^{1+s_k})L_j + Op(S^{1+s_k})$$

$$Q^*_{k-1}A = AQ_{k-1} + Op(S^{s_k}) = \langle D\rangle^{s_k - s_{k-1}} Op(S^{s_{k-1}})Q_{k-1} + Op(S^{s_k}).$$

The worst term is

$$(Q^*_{k-1}APu, AQ_{k-1}u) = (\underbrace{\langle D\rangle^{s_k-s_{k-1}}AQ_{k-1}}_{\text{of order } 2s_k - s_{k-1}+1=0} Pu, \underbrace{Op(S^{s_{k-1}})}_{H^{s_{k-1}}\to L^2} Q_{k-1}u)$$

The other terms and $(Q_{k-1}u, P'Au)$ can be treated similarly. This completes the induction.

Proof: estimate on $i[Q, Q']$ with $Q \in \mathcal{Q}_{k-2}$ and $Q' = P''$

It remains to estimate $(Q_{k-1}u, P'Au)$ and $(P'u, Q^*_{k-1}Au)$. First

$$|(P'u, Q^*_{k-1}Au)| \lesssim \left(\mathrm{Re}(Pu, u)\right)^{\frac{1}{2}} \left(\mathrm{Re}(PQ^*_{k-1}Au, Q^*_{k-1}Au)\right)^{\frac{1}{2}}$$

using again the P'-Cauchy-Schwarz inequality. Then, use

$$PQ^*_{k-1}A = \underbrace{Q^*_{k-1}A}_{\in Op(S^{1+s_k})} P + \sum_1^r Op(S^{1+s_k})L_j + Op(S^{1+s_k})$$

$$Q^*_{k-1}A = AQ_{k-1} + Op(S^{s_k}) = \langle D\rangle^{s_k - s_{k-1}} Op(S^{s_{k-1}})Q_{k-1} + Op(S^{s_k}).$$

The worst term is

$$(Q^*_{k-1}APu, AQ_{k-1}u) = \Big(\underbrace{\langle D\rangle^{s_k - s_{k-1}} AQ_{k-1}}_{\text{of order } 2s_k - s_{k-1} + 1 = 0} Pu, \underbrace{Op(S^{s_{k-1}})}_{H^{s_{k-1}} \to L^2} Q_{k-1}u\Big)$$

The other terms and $(Q_{k-1}u, P'Au)$ can be treated similarly.
This completes the induction.

Proof: estimate on $i[Q, Q']$ with $Q \in \mathcal{Q}_{k-2}$ and $Q' = P''$

It remains to estimate $(Q_{k-1}u, P'Au)$ and $(P'u, Q^*_{k-1}Au)$. First

$$|(P'u, Q^*_{k-1}Au)| \lesssim \big(\mathrm{Re}(Pu,u)\big)^{\frac{1}{2}} \big(\mathrm{Re}(PQ^*_{k-1}Au, Q^*_{k-1}Au)\big)^{\frac{1}{2}}$$

using again the P'-Cauchy-Schwarz inequality. Then, use

$$PQ^*_{k-1}A = \underbrace{Q^*_{k-1}A}_{\in Op(S^{1+s_k})} P + \sum_1^r Op(S^{1+s_k})L_j + Op(S^{1+s_k})$$

$$Q^*_{k-1}A = AQ_{k-1} + Op(S^{s_k}) = \langle D\rangle^{s_k - s_{k-1}} Op(S^{s_{k-1}})Q_{k-1} + Op(S^{s_k}).$$

The worst term is

$$(Q^*_{k-1}APu, AQ_{k-1}u) = \Big(\underbrace{\langle D\rangle^{s_k - s_{k-1}} AQ_{k-1}}_{\text{of order } 2s_k - s_{k-1}+1=0} Pu, \underbrace{Op(S^{s_{k-1}})}_{H^{s_{k-1}} \to L^2} Q_{k-1}u\Big)$$

The other terms and $(Q_{k-1}u, P'Au)$ can be treated similarly. This completes the induction.

Proof: estimate on $i[Q, Q']$ with $Q \in \mathcal{Q}_{k-2}$ and $Q' = P''$

It remains to estimate $(Q_{k-1}u, P'Au)$ and $(P'u, Q^*_{k-1}Au)$. First

$$|(P'u, Q^*_{k-1}Au)| \lesssim \big(\mathrm{Re}(Pu, u)\big)^{\frac{1}{2}} \big(\mathrm{Re}(PQ^*_{k-1}Au, Q^*_{k-1}Au)\big)^{\frac{1}{2}}$$

using again the P'-Cauchy-Schwarz inequality. Then, use

$$PQ^*_{k-1}A = \underbrace{Q^*_{k-1}A}_{\in Op(S^{1+s_k})} P + \sum_1^r Op(S^{1+s_k})L_j + Op(S^{1+s_k})$$

$$Q^*_{k-1}A = AQ_{k-1} + Op(S^{s_k}) = \langle D\rangle^{s_k - s_{k-1}} Op(S^{s_{k-1}})Q_{k-1} + Op(S^{s_k}).$$

The worst term is

$$(Q^*_{k-1}APu, AQ_{k-1}u) = \Big(\underbrace{\langle D\rangle^{s_k - s_{k-1}} AQ_{k-1}}_{\text{of order } 2s_k - s_{k-1} + 1 = 0} Pu, \underbrace{Op(S^{s_{k-1}})}_{H^{s_{k-1}} \to L^2} Q_{k-1}u\Big)$$

The other terms and $(Q_{k-1}u, P'Au)$ can be treated similarly. This completes the induction.

Proof: estimate on $i[Q, Q']$ with $Q \in \mathcal{Q}_{k-2}$ and $Q' = P''$

It remains to estimate $(Q_{k-1}u, P'Au)$ and $(P'u, Q^*_{k-1}Au)$. First

$$|(P'u, Q^*_{k-1}Au)| \lesssim \left(\mathrm{Re}(Pu,u)\right)^{\frac{1}{2}} \left(\mathrm{Re}(PQ^*_{k-1}Au, Q^*_{k-1}Au)\right)^{\frac{1}{2}}$$

using again the P'-Cauchy-Schwarz inequality. Then, use

$$PQ^*_{k-1}A = \underbrace{Q^*_{k-1}A}_{\in Op(S^{1+s_k})} P + \sum_1^r Op(S^{1+s_k})L_j + Op(S^{1+s_k})$$

$$Q^*_{k-1}A = AQ_{k-1} + Op(S^{s_k}) = \langle D\rangle^{s_k - s_{k-1}} Op(S^{s_{k-1}})Q_{k-1} + Op(S^{s_k}).$$

The worst term is

$$(Q^*_{k-1}APu, AQ_{k-1}u) = (\underbrace{\langle D\rangle^{s_k - s_{k-1}} AQ_{k-1}}_{\text{of order } 2s_k - s_{k-1} + 1 = 0} Pu, \underbrace{Op(S^{s_{k-1}})}_{H^{s_{k-1}} \to L^2} Q_{k-1}u)$$

The other terms and $(Q_{k-1}u, P'Au)$ can be treated similarly. This completes the induction.

Proof: estimate on $i[Q, Q']$ with $Q \in \mathcal{Q}_{k-2}$ and $Q' = P''$

It remains to estimate $(Q_{k-1}u, P'Au)$ and $(P'u, Q_{k-1}^* Au)$. First

$$|(P'u, Q_{k-1}^* Au)| \lesssim \left(\mathrm{Re}(Pu, u)\right)^{\frac{1}{2}} \left(\mathrm{Re}(PQ_{k-1}^* Au, Q_{k-1}^* Au)\right)^{\frac{1}{2}}$$

using again the P'-Cauchy-Schwarz inequality. Then, use

$$PQ_{k-1}^* A = \underbrace{Q_{k-1}^* A}_{\in Op(S^{1+s_k})} P + \sum_1^r Op(S^{1+s_k}) L_j + Op(S^{1+s_k})$$

$$Q_{k-1}^* A = AQ_{k-1} + Op(S^{s_k}) = \langle D \rangle^{s_k - s_{k-1}} Op(S^{s_{k-1}}) Q_{k-1} + Op(S^{s_k}).$$

The worst term is

$$(Q_{k-1}^* APu, AQ_{k-1}u) = \Big(\underbrace{\langle D \rangle^{s_k - s_{k-1}} AQ_{k-1}}_{\text{of order } 2s_k - s_{k-1} + 1 = 0} Pu, \underbrace{Op(S^{s_{k-1}})}_{H^{s_{k-1}} \to L^2} Q_{k-1}u\Big)$$

The other terms and $(Q_{k-1}u, P'Au)$ can be treated similarly. This completes the induction.

Proof: estimate on $i[Q, Q']$ with $Q \in \mathcal{Q}_{k-2}$ and $Q' = P''$

It remains to estimate $(Q_{k-1}u, P'Au)$ and $(P'u, Q^*_{k-1}Au)$. First

$$|(P'u, Q^*_{k-1}Au)| \lesssim \big(\mathrm{Re}(Pu, u)\big)^{\frac{1}{2}} \big(\mathrm{Re}(PQ^*_{k-1}Au, Q^*_{k-1}Au)\big)^{\frac{1}{2}}$$

using again the P'-Cauchy-Schwarz inequality. Then, use

$$PQ^*_{k-1}A = \underbrace{Q^*_{k-1}A}_{\in Op(S^{1+s_k})} P + \sum_1^r Op(S^{1+s_k})L_j + Op(S^{1+s_k})$$

$$Q^*_{k-1}A = AQ_{k-1} + Op(S^{s_k}) = \langle D\rangle^{s_k - s_{k-1}} Op(S^{s_{k-1}})Q_{k-1} + Op(S^{s_k}).$$

The worst term is

$$(Q^*_{k-1}APu, AQ_{k-1}u) = \Big(\underbrace{\langle D\rangle^{s_k - s_{k-1}} AQ_{k-1}}_{\text{of order } 2s_k - s_{k-1} + 1 = 0} Pu, \underbrace{Op(S^{s_{k-1}})}_{H^{s_{k-1}} \to L^2} Q_{k-1}u\Big)$$

The other terms and $(Q_{k-1}u, P'Au)$ can be treated similarly.
This completes the induction.

Proof: estimate on $i[Q, Q']$ with $Q \in \mathcal{Q}_{k-2}$ and $Q' = P''$

It remains to estimate $(Q_{k-1}u, P'Au)$ and $(P'u, Q_{k-1}^*Au)$. First

$$|(P'u, Q_{k-1}^*Au)| \lesssim \big(\mathrm{Re}(Pu, u)\big)^{\frac{1}{2}} \big(\mathrm{Re}(PQ_{k-1}^*Au, Q_{k-1}^*Au)\big)^{\frac{1}{2}}$$

using again the P'-Cauchy-Schwarz inequality. Then, use

$$PQ_{k-1}^*A = \underbrace{Q_{k-1}^*A}_{\in Op(S^{1+s_k})} P + \sum_1^r Op(S^{1+s_k})L_j + Op(S^{1+s_k})$$

$$Q_{k-1}^*A = AQ_{k-1} + Op(S^{s_k}) = \langle D\rangle^{s_k - s_{k-1}} Op(S^{s_{k-1}})Q_{k-1} + Op(S^{s_k}).$$

The worst term is

$$(Q_{k-1}^*APu, AQ_{k-1}u) = \Big(\underbrace{\langle D\rangle^{s_k - s_{k-1}} AQ_{k-1}}_{\text{of order } 2s_k - s_{k-1} + 1 = 0} Pu, \underbrace{Op(S^{s_{k-1}})}_{H^{s_{k-1}} \to L^2} Q_{k-1}u\Big)$$

The other terms and $(Q_{k-1}u, P'Au)$ can be treated similarly. This completes the induction.

Proof: estimate on $i[Q, Q']$ with $Q \in \mathcal{Q}_{k-2}$ and $Q' = P''$

It remains to estimate $(Q_{k-1}u, P'Au)$ and $(P'u, Q_{k-1}^*Au)$. First

$$|(P'u, Q_{k-1}^*Au)| \lesssim \big(\mathrm{Re}(Pu,u)\big)^{\frac{1}{2}} \big(\mathrm{Re}(PQ_{k-1}^*Au, Q_{k-1}^*Au)\big)^{\frac{1}{2}}$$

using again the P'-Cauchy-Schwarz inequality. Then, use

$$PQ_{k-1}^*A = \underbrace{Q_{k-1}^*A}_{\in Op(S^{1+s_k})} P + \sum_1^r Op(S^{1+s_k})L_j + Op(S^{1+s_k})$$

$$Q_{k-1}^*A = AQ_{k-1} + Op(S^{s_k}) = \langle D\rangle^{s_k - s_{k-1}} Op(S^{s_{k-1}})Q_{k-1} + Op(S^{s_k}).$$

The worst term is

$$(Q_{k-1}^*APu, AQ_{k-1}u) = \Big(\underbrace{\langle D\rangle^{s_k - s_{k-1}} AQ_{k-1}}_{\text{of order } 2s_k - s_{k-1} + 1 = 0} Pu, \underbrace{Op(S^{s_{k-1}})}_{H^{s_{k-1}} \to L^2} Q_{k-1}u\Big)$$

The other terms and $(Q_{k-1}u, P'Au)$ can be treated similarly. This completes the induction.

Proof: estimate on $i[Q, Q']$ with $Q \in \mathcal{Q}_{k-2}$ and $Q' = P''$

It remains to estimate $(Q_{k-1}u, P'Au)$ and $(P'u, Q_{k-1}^*Au)$. First

$$|(P'u, Q_{k-1}^*Au)| \lesssim \left(\mathrm{Re}(Pu, u)\right)^{\frac{1}{2}}\left(\mathrm{Re}(PQ_{k-1}^*Au, Q_{k-1}^*Au)\right)^{\frac{1}{2}}$$

using again the P'-Cauchy-Schwarz inequality. Then, use

$$PQ_{k-1}^*A = \underbrace{Q_{k-1}^*A}_{\in Op(S^{1+s_k})} P + \sum_1^r Op(S^{1+s_k})L_j + Op(S^{1+s_k})$$

$$Q_{k-1}^*A = AQ_{k-1} + Op(S^{s_k}) = \langle D\rangle^{s_k - s_{k-1}} Op(S^{s_{k-1}})Q_{k-1} + Op(S^{s_k}).$$

The worst term is

$$(Q_{k-1}^*APu, AQ_{k-1}u) = \Big(\underbrace{\langle D\rangle^{s_k - s_{k-1}} AQ_{k-1}}_{\text{of order } 2s_k - s_{k-1} + 1 = 0} Pu, \underbrace{Op(S^{s_{k-1}})}_{H^{s_{k-1}} \to L^2} Q_{k-1}u\Big)$$

The other terms and $(Q_{k-1}u, P'Au)$ can be treated similarly. This completes the induction.

Proof: estimate on $i[Q, Q']$ with $Q \in \mathcal{Q}_{k-2}$ and $Q' = P''$

It remains to estimate $(Q_{k-1}u, P'Au)$ and $(P'u, Q^*_{k-1}Au)$. First

$$|(P'u, Q^*_{k-1}Au)| \lesssim \left(\mathrm{Re}(Pu, u)\right)^{\frac{1}{2}} \left(\mathrm{Re}(PQ^*_{k-1}Au, Q^*_{k-1}Au)\right)^{\frac{1}{2}}$$

using again the P'-Cauchy-Schwarz inequality. Then, use

$$PQ^*_{k-1}A = \underbrace{Q^*_{k-1}A}_{\in Op(S^{1+s_k})} P + \sum_1^r Op(S^{1+s_k})L_j + Op(S^{1+s_k})$$

$$Q^*_{k-1}A = AQ_{k-1} + Op(S^{s_k}) = \langle D\rangle^{s_k - s_{k-1}} Op(S^{s_{k-1}})Q_{k-1} + Op(S^{s_k}).$$

The worst term is

$$(Q^*_{k-1}APu, AQ_{k-1}u) = (\underbrace{\langle D\rangle^{s_k - s_{k-1}} AQ_{k-1}}_{\text{of order } 2s_k - s_{k-1}+1=0} Pu, \underbrace{Op(S^{s_{k-1}})}_{H^{s_{k-1}} \to L^2} Q_{k-1}u)$$

The other terms and $(Q_{k-1}u, P'Au)$ can be treated similarly. This completes the induction.

Proof: estimate on $i[Q, Q']$ with $Q \in \mathcal{Q}_{k-2}$ and $Q' = P''$

It remains to estimate $(Q_{k-1}u, P'Au)$ and $(P'u, Q^*_{k-1}Au)$. First

$$|(P'u, Q^*_{k-1}Au)| \lesssim \left(\mathrm{Re}(Pu,u)\right)^{\frac{1}{2}} \left(\mathrm{Re}(PQ^*_{k-1}Au, Q^*_{k-1}Au)\right)^{\frac{1}{2}}$$

using again the P'-Cauchy-Schwarz inequality. Then, use

$$\begin{aligned} PQ^*_{k-1}A &= \underbrace{Q^*_{k-1}A}_{\in Op(S^{1+s_k})} P + \sum_1^r Op(S^{1+s_k})L_j + Op(S^{1+s_k}) \\ Q^*_{k-1}A &= AQ_{k-1} + Op(S^{s_k}) = \langle D\rangle^{s_k - s_{k-1}} Op(S^{s_{k-1}})Q_{k-1} + Op(S^{s_k}). \end{aligned}$$

The worst term is

$$(Q^*_{k-1}APu, AQ_{k-1}u) = \Big(\underbrace{\langle D\rangle^{s_k - s_{k-1}} AQ_{k-1}}_{\text{of order } 2s_k - s_{k-1} + 1 = 0} Pu, \underbrace{Op(S^{s_{k-1}})}_{H^{s_{k-1}} \to L^2} Q_{k-1}u\Big)$$

The other terms and $(Q_{k-1}u, P'Au)$ can be treated similarly. This completes the induction.

Proof: estimate on $i[Q, Q']$ with $Q \in \mathcal{Q}_{k-2}$ and $Q' = P''$

It remains to estimate $(Q_{k-1}u, P'Au)$ and $(P'u, Q^*_{k-1}Au)$. First

$$|(P'u, Q^*_{k-1}Au)| \lesssim \left(\mathrm{Re}(Pu, u)\right)^{\frac{1}{2}} \left(\mathrm{Re}(PQ^*_{k-1}Au, Q^*_{k-1}Au)\right)^{\frac{1}{2}}$$

using again the P'-Cauchy-Schwarz inequality. Then, use

$$\begin{aligned} PQ^*_{k-1}A &= \underbrace{Q^*_{k-1}A}_{\in Op(S^{1+s_k})} P + \sum_1^r Op(S^{1+s_k})L_j + Op(S^{1+s_k}) \\ Q^*_{k-1}A &= AQ_{k-1} + Op(S^{s_k}) = \langle D\rangle^{s_k - s_{k-1}} Op(S^{s_{k-1}})Q_{k-1} + Op(S^{s_k}). \end{aligned}$$

The worst term is

$$(Q^*_{k-1}APu, AQ_{k-1}u) = \Big(\underbrace{\langle D\rangle^{s_k - s_{k-1}} AQ_{k-1}}_{\text{of order } 2s_k - s_{k-1} + 1 = 0} Pu, \underbrace{Op(S^{s_{k-1}})}_{H^{s_{k-1}} \to L^2} Q_{k-1}u \Big)$$

The other terms and $(Q_{k-1}u, P'Au)$ can be treated similarly. This completes the induction.

Proof: estimate on $i[Q, Q']$ with $Q \in \mathcal{Q}_{k-2}$ and $Q' = P''$

It remains to estimate $(Q_{k-1}u, P'Au)$ and $(P'u, Q^*_{k-1}Au)$. First

$$|(P'u, Q^*_{k-1}Au)| \lesssim \left(\mathrm{Re}(Pu, u)\right)^{\frac{1}{2}} \left(\mathrm{Re}(PQ^*_{k-1}Au, Q^*_{k-1}Au)\right)^{\frac{1}{2}}$$

using again the P'-Cauchy-Schwarz inequality. Then, use

$$\begin{aligned} PQ^*_{k-1}A &= \underbrace{Q^*_{k-1}A}_{\in Op(S^{1+s_k})} P + \sum_1^r Op(S^{1+s_k})L_j + Op(S^{1+s_k}) \\ Q^*_{k-1}A &= AQ_{k-1} + Op(S^{s_k}) = \langle D\rangle^{s_k - s_{k-1}} Op(S^{s_{k-1}})Q_{k-1} + Op(S^{s_k}). \end{aligned}$$

The worst term is

$$(Q^*_{k-1}APu, AQ_{k-1}u) = \Big(\underbrace{\langle D\rangle^{s_k - s_{k-1}} AQ_{k-1}}_{\text{of order } 2s_k - s_{k-1} + 1 = 0} Pu, \underbrace{Op(S^{s_{k-1}})}_{H^{s_{k-1}} \to L^2} Q_{k-1}u\Big)$$

The other terms and $(Q_{k-1}u, P'Au)$ can be treated similarly.

This completes the induction.

Proof: estimate on $i[Q, Q']$ with $Q \in \mathcal{Q}_{k-2}$ and $Q' = P''$

It remains to estimate $(Q_{k-1}u, P'Au)$ and $(P'u, Q^*_{k-1}Au)$. First

$$|(P'u, Q^*_{k-1}Au)| \lesssim \big(\mathrm{Re}(Pu, u)\big)^{\frac{1}{2}} \big(\mathrm{Re}(PQ^*_{k-1}Au, Q^*_{k-1}Au)\big)^{\frac{1}{2}}$$

using again the P'-Cauchy-Schwarz inequality. Then, use

$$\begin{aligned} PQ^*_{k-1}A &= \underbrace{Q^*_{k-1}A}_{\in Op(S^{1+s_k})} P + \sum_1^r Op(S^{1+s_k})L_j + Op(S^{1+s_k}) \\ Q^*_{k-1}A &= AQ_{k-1} + Op(S^{s_k}) = \langle D\rangle^{s_k - s_{k-1}} Op(S^{s_{k-1}})Q_{k-1} + Op(S^{s_k}). \end{aligned}$$

The worst term is

$$(Q^*_{k-1}APu, AQ_{k-1}u) = \Big(\underbrace{\langle D\rangle^{s_k - s_{k-1}} AQ_{k-1}}_{\text{of order } 2s_k - s_{k-1} + 1 = 0} Pu, \underbrace{Op(S^{s_{k-1}})}_{H^{s_{k-1}} \to L^2} Q_{k-1}u\Big)$$

The other terms and $(Q_{k-1}u, P'Au)$ can be treated similarly. This completes the induction.

Conclusion of the proof

Denote by $(V_\ell)_{1\le \ell \le M}$ the (finite) collection of vector fields

$$\{L_k, [L_{k_1}, L_{k_2}], \dots, [L_{k_1}, [L_{k_2}, \dots [L_{k_{N-1}}, L_{k_N}]\}.$$

Near any given x_0, one can find n of them $V_{\ell_1}, \dots, V_{\ell_n}$ of rank n (as vectors of $T_x\mathbb{R}^n$, x close to x_0) and a matrix of smooth functions $(a_{ij}(x))_{1\le i,j\le n}$ such that

$$\frac{\partial}{\partial x_i} = \sum_{j=1}^{n} a_{ij}(x) V_{\ell_j}, \qquad |x - x_0| \ll 1.$$

If $\chi \in C_0^\infty$ is supported near x_0, we infer

$$||\chi \partial_{x_i} u||_{(\epsilon-1)} \lesssim \sum_{j=1}^{n} ||V_{\ell_j} u||_{(\epsilon-1)} \lesssim ||Pu|| + ||u||$$

hence

$$||\chi u||_{(\epsilon)} \lesssim \sum ||\partial_{x_i}(\chi u)||_{(\epsilon-1)} + ||u||_{(0)} \lesssim \underbrace{||(\nabla \chi) u||_{(\epsilon-1)}}_{\lesssim ||u||_{(0)} = ||u||} + ||Pu|| + ||u|| \ \square$$

Conclusion of the proof

Denote by $(V_\ell)_{1\le \ell\le M}$ the (finite) collection of vector fields

$$\{L_k, [L_{k_1}, L_{k_2}], \dots, [L_{k_1}, [L_{k_2}, \dots [L_{k_{N-1}}, L_{k_N}]\}\,.$$

Near any given x_0, one can find n of them $V_{\ell_1}, \dots, V_{\ell_n}$ of rank n (as vectors of $T_x\mathbb{R}^n$, x close to x_0) and a matrix of smooth functions $(a_{ij}(x))_{1\le i,j\le n}$ such that

$$\frac{\partial}{\partial x_i} = \sum_{j=1}^{n} a_{ij}(x) V_{\ell_j}, \qquad |x-x_0| \ll 1.$$

If $\chi \in C_0^\infty$ is supported near x_0, we infer

$$||\chi \partial_{x_i} u||_{(\epsilon-1)} \lesssim \sum_{j=1}^{n} ||V_{\ell_j} u||_{(\epsilon-1)} \lesssim ||Pu|| + ||u||$$

hence

$$||\chi u||_{(\epsilon)} \lesssim \sum ||\partial_{x_i}(\chi u)||_{(\epsilon-1)} + ||u||_{(0)} \lesssim \underbrace{||(\nabla\chi)u||_{(\epsilon-1)}}_{\lesssim ||u||_{(0)} = ||u||} + ||Pu|| + ||u|| \;\Box$$

Conclusion of the proof

Denote by $(V_\ell)_{1\le\ell\le M}$ the (finite) collection of vector fields

$$\left\{L_k, [L_{k_1}, L_{k_2}], \ldots, [L_{k_1}, [L_{k_2}, \ldots [L_{k_{N-1}}, L_{k_N}]\right\}.$$

Near any given x_0, one can find n of them $V_{\ell_1}, \ldots, V_{\ell_n}$ of rank n (as vectors of $T_x\mathbb{R}^n$, x close to x_0) and a matrix of smooth functions $(a_{ij}(x))_{1\le i,j\le n}$ such that

$$\frac{\partial}{\partial x_i} = \sum_{j=1}^{n} a_{ij}(x) V_{\ell_j}, \qquad |x - x_0| \ll 1.$$

If $\chi \in C_0^\infty$ is supported near x_0, we infer

$$||\chi \partial_{x_i} u||_{(\epsilon-1)} \lesssim \sum_{j=1}^{n} ||V_{\ell_j} u||_{(\epsilon-1)} \lesssim ||Pu|| + ||u||$$

hence

$$||\chi u||_{(\epsilon)} \lesssim \sum ||\partial_{x_i}(\chi u)||_{(\epsilon-1)} + ||u||_{(0)} \lesssim \underbrace{||(\nabla\chi)u||_{(\epsilon-1)}}_{\lesssim ||u||_{(0)} = ||u||} + ||Pu|| + ||u|| \; \square$$

Conclusion of the proof

Denote by $(V_\ell)_{1\le \ell\le M}$ the (finite) collection of vector fields

$$\left\{L_k,[L_{k_1},L_{k_2}],\dots,[L_{k_1},[L_{k_2},...[L_{k_{N-1}},L_{k_N}]\right\}.$$

Near any given x_0, one can find n of them $V_{\ell_1},\dots,V_{\ell_n}$ of rank n (as vectors of $T_x\mathbb{R}^n$, x close to x_0) and a matrix of smooth functions $(a_{ij}(x))_{1\le i,j\le n}$ such that

$$\frac{\partial}{\partial x_i}=\sum_{j=1}^n a_{ij}(x)V_{\ell_j},\qquad |x-x_0|\ll 1.$$

If $\chi\in C_0^\infty$ is supported near x_0, we infer

$$||\chi\partial_{x_i}u||_{(\epsilon-1)}\lesssim\sum_{j=1}^n||V_{\ell_j}u||_{(\epsilon-1)}\lesssim||Pu||+||u||$$

hence

$$||\chi u||_{(\epsilon)}\lesssim\sum||\partial_{x_i}(\chi u)||_{(\epsilon-1)}+||u||_{(0)}\lesssim\underbrace{||(\nabla\chi)u||_{(\epsilon-1)}}_{\lesssim||u||_{(0)}=||u||}+||Pu||+||u||\ \square$$

Conclusion of the proof

Denote by $(V_\ell)_{1\leq \ell\leq M}$ the (finite) collection of vector fields

$$\{L_k, [L_{k_1}, L_{k_2}], \dots, [L_{k_1}, [L_{k_2}, ...[L_{k_{N-1}}, L_{k_N}]\}\,.$$

Near any given x_0, one can find n of them $V_{\ell_1}, \dots, V_{\ell_n}$ of rank n (as vectors of $T_x\mathbb{R}^n$, x close to x_0) and a matrix of smooth functions $(a_{ij}(x))_{1\leq i,j\leq n}$ such that

$$\frac{\partial}{\partial x_i} = \sum_{j=1}^{n} a_{ij}(x) V_{\ell_j}, \qquad |x - x_0| \ll 1.$$

If $\chi \in C_0^\infty$ is supported near x_0, we infer

$$||\chi\partial_{x_i} u||_{(\epsilon-1)} \lesssim \sum_{j=1}^{n} ||V_{\ell_j} u||_{(\epsilon-1)} \lesssim ||Pu|| + ||u||$$

hence

$$||\chi u||_{(\epsilon)} \lesssim \sum ||\partial_{x_i}(\chi u)||_{(\epsilon-1)} + ||u||_{(0)} \lesssim \underbrace{||(\nabla\chi)u||_{(\epsilon-1)}}_{\lesssim ||u||_{(0)} = ||u||} + ||Pu|| + ||u|| \;\square$$

ΨDO (part 1): Proof of the L^2 Compactness

We let $h = 1$. Pick $a_\epsilon \in C_0^\infty$ such that $a_\epsilon \to a$ in $S^{0,0}$. Observe that

$$
\begin{aligned}
Op_h(a_\epsilon)u(x) &= (2\pi)^{-n} \int\int e^{i(x-y)\cdot\xi} a_\epsilon(x,\xi) u(y) dy d\xi \\
&= \int K_\epsilon(x,y) u(y) dy
\end{aligned}
$$

where

$$
\begin{aligned}
K_\epsilon(x,y) &= (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} a_\epsilon(x,\xi) d\xi \\
&= (2\pi)^{-n} \widehat{a_\epsilon}(x, y-x)
\end{aligned}
$$

belongs to $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n) \subset L^2(\mathbb{R}^n \times \mathbb{R}^n)$. Thus $Op_h(a_\epsilon)$ is Hilbert-Schmidt hence compact. We conclude with

$$||Op_h(a) - Op_h(a_\epsilon)||_{L^2 \to L^2} \le C||a - a_\epsilon||_{S^{0,0},N} \to 0.$$

ΨDO (part 1): Proof of the L^2 Compactness

We let $h = 1$. Pick $a_\epsilon \in C_0^\infty$ such that $a_\epsilon \to a$ in $S^{0,0}$. Observe that

$$\begin{aligned} Op_h(a_\epsilon)u(x) &= (2\pi)^{-n} \int\int e^{i(x-y)\cdot\xi} a_\epsilon(x,\xi)u(y)dyd\xi \\ &= \int K_\epsilon(x,y)u(y)dy \end{aligned}$$

where

$$\begin{aligned} K_\epsilon(x,y) &= (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} a_\epsilon(x,\xi)d\xi \\ &= (2\pi)^{-n} \widehat{a_\epsilon}(x, y-x) \end{aligned}$$

belongs to $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n) \subset L^2(\mathbb{R}^n \times \mathbb{R}^n)$. Thus $Op_h(a_\epsilon)$ is Hilbert-Schmidt hence compact. We conclude with

$$||Op_h(a) - Op_h(a_\epsilon)||_{L^2 \to L^2} \le C||a - a_\epsilon||_{S^{0,0},N} \to 0.$$

ΨDO (part 1): Proof of the L^2 Compactness

We let $h = 1$. Pick $a_\epsilon \in C_0^\infty$ such that $a_\epsilon \to a$ in $S^{0,0}$. Observe that

$$\begin{aligned} Op_h(a_\epsilon)u(x) &= (2\pi)^{-n} \int\int e^{i(x-y)\cdot\xi} a_\epsilon(x,\xi)u(y)dyd\xi \\ &= \int K_\epsilon(x,y)u(y)dy \end{aligned}$$

where

$$\begin{aligned} K_\epsilon(x,y) &= (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} a_\epsilon(x,\xi)d\xi \\ &= (2\pi)^{-n} \widehat{a_\epsilon}(x, y-x) \end{aligned}$$

belongs to $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n) \subset L^2(\mathbb{R}^n \times \mathbb{R}^n)$. Thus $Op_h(a_\epsilon)$ is Hilbert-Schmidt hence compact. We conclude with

$$||Op_h(a) - Op_h(a_\epsilon)||_{L^2 \to L^2} \le C||a - a_\epsilon||_{S^{0,0},N} \to 0.$$

ΨDO (part 1): Proof of the L^2 Compactness

We let $h = 1$. Pick $a_\epsilon \in C_0^\infty$ such that $a_\epsilon \to a$ in $S^{0,0}$. Observe that

$$Op_h(a_\epsilon)u(x) = (2\pi)^{-n} \int\int e^{i(x-y)\cdot\xi} a_\epsilon(x,\xi)u(y)dyd\xi$$
$$= \int K_\epsilon(x,y)u(y)dy$$

where

$$K_\epsilon(x,y) = (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} a_\epsilon(x,\xi)d\xi$$
$$= (2\pi)^{-n} \widehat{a_\epsilon}(x, y-x)$$

belongs to $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n) \subset L^2(\mathbb{R}^n \times \mathbb{R}^n)$. Thus $Op_h(a_\epsilon)$ is Hilbert-Schmidt hence compact. We conclude with

$$||Op_h(a) - Op_h(a_\epsilon)||_{L^2 \to L^2} \leq C||a - a_\epsilon||_{S^{0,0},N} \to 0.$$

ΨDO (part 1): Proof of the L^2 Compactness

We let $h = 1$. Pick $a_\epsilon \in C_0^\infty$ such that $a_\epsilon \to a$ in $S^{0,0}$. Observe that

$$\begin{aligned} Op_h(a_\epsilon)u(x) &= (2\pi)^{-n} \int\int e^{i(x-y)\cdot\xi} a_\epsilon(x,\xi)u(y)dyd\xi \\ &= \int K_\epsilon(x,y)u(y)dy \end{aligned}$$

where

$$\begin{aligned} K_\epsilon(x,y) &= (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} a_\epsilon(x,\xi)d\xi \\ &= (2\pi)^{-n} \widehat{a_\epsilon}(x, y-x) \end{aligned}$$

belongs to $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n) \subset L^2(\mathbb{R}^n \times \mathbb{R}^n)$. Thus $Op_h(a_\epsilon)$ is Hilbert-Schmidt hence compact. We conclude with

$$||Op_h(a) - Op_h(a_\epsilon)||_{L^2 \to L^2} \leq C||a - a_\epsilon||_{S^{0,0},N} \to 0.$$

ΨDO (part 1): Proof of the L^2 Compactness

We let $h = 1$. Pick $a_\epsilon \in C_0^\infty$ such that $a_\epsilon \to a$ in $S^{0,0}$. Observe that

$$\begin{aligned} Op_h(a_\epsilon)u(x) &= (2\pi)^{-n} \int\int e^{i(x-y)\cdot\xi} a_\epsilon(x,\xi)u(y)dyd\xi \\ &= \int K_\epsilon(x,y)u(y)dy \end{aligned}$$

where

$$\begin{aligned} K_\epsilon(x,y) &= (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} a_\epsilon(x,\xi)d\xi \\ &= (2\pi)^{-n} \widehat{a_\epsilon}(x, y-x) \end{aligned}$$

belongs to $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n) \subset L^2(\mathbb{R}^n \times \mathbb{R}^n)$. Thus $Op_h(a_\epsilon)$ is Hilbert-Schmidt hence compact. We conclude with

$$||Op_h(a) - Op_h(a_\epsilon)||_{L^2 \to L^2} \leq C||a - a_\epsilon||_{S^{0,0},N} \to 0.$$

ΨDO (part 1): Proof of the L^2 Compactness

We let $h = 1$. Pick $a_\epsilon \in C_0^\infty$ such that $a_\epsilon \to a$ in $S^{0,0}$. Observe that

$$\begin{aligned} Op_h(a_\epsilon)u(x) &= (2\pi)^{-n} \int \int e^{i(x-y)\cdot\xi} a_\epsilon(x,\xi)u(y)dyd\xi \\ &= \int K_\epsilon(x,y)u(y)dy \end{aligned}$$

where

$$\begin{aligned} K_\epsilon(x,y) &= (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} a_\epsilon(x,\xi)d\xi \\ &= (2\pi)^{-n} \widehat{a_\epsilon}(x, y-x) \end{aligned}$$

belongs to $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n) \subset L^2(\mathbb{R}^n \times \mathbb{R}^n)$. Thus $Op_h(a_\epsilon)$ is Hilbert-Schmidt hence compact. We conclude with

$$||Op_h(a) - Op_h(a_\epsilon)||_{L^2 \to L^2} \leq C||a - a_\epsilon||_{S^{0,0},N} \to 0.$$

ΨDO (part 1): Proof of the L^2 Compactness

We let $h = 1$. Pick $a_\epsilon \in C_0^\infty$ such that $a_\epsilon \to a$ in $S^{0,0}$. Observe that

$$\begin{aligned} Op_h(a_\epsilon)u(x) &= (2\pi)^{-n} \int\int e^{i(x-y)\cdot\xi} a_\epsilon(x,\xi)u(y)dyd\xi \\ &= \int K_\epsilon(x,y)u(y)dy \end{aligned}$$

where

$$\begin{aligned} K_\epsilon(x,y) &= (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} a_\epsilon(x,\xi)d\xi \\ &= (2\pi)^{-n} \widehat{a_\epsilon}(x, y-x) \end{aligned}$$

belongs to $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n) \subset L^2(\mathbb{R}^n \times \mathbb{R}^n)$. Thus $Op_h(a_\epsilon)$ is Hilbert-Schmidt hence compact. We conclude with

$$||Op_h(a) - Op_h(a_\epsilon)||_{L^2 \to L^2} \leq C||a - a_\epsilon||_{S^{0,0},N} \to 0.$$

ΨDO (part 1): Proof of the L^2 Compactness

We let $h = 1$. Pick $a_\epsilon \in C_0^\infty$ such that $a_\epsilon \to a$ in $S^{0,0}$. Observe that

$$
\begin{aligned}
Op_h(a_\epsilon)u(x) &= (2\pi)^{-n} \int\int e^{i(x-y)\cdot\xi} a_\epsilon(x,\xi)u(y)dyd\xi \\
&= \int K_\epsilon(x,y)u(y)dy
\end{aligned}
$$

where

$$
\begin{aligned}
K_\epsilon(x,y) &= (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} a_\epsilon(x,\xi)d\xi \\
&= (2\pi)^{-n} \widehat{a_\epsilon}(x, y-x)
\end{aligned}
$$

belongs to $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n) \subset L^2(\mathbb{R}^n \times \mathbb{R}^n)$. Thus $Op_h(a_\epsilon)$ is Hilbert-Schmidt hence compact. We conclude with

$$
||Op_h(a) - Op_h(a_\epsilon)||_{L^2 \to L^2} \leq C||a - a_\epsilon||_{S^{0,0},N} \to 0.
$$

ΨDO (part 1): Proof of the L^2 Compactness

We let $h = 1$. Pick $a_\epsilon \in C_0^\infty$ such that $a_\epsilon \to a$ in $S^{0,0}$. Observe that

$$Op_h(a_\epsilon)u(x) = (2\pi)^{-n} \int\int e^{i(x-y)\cdot\xi} a_\epsilon(x,\xi)u(y)dyd\xi$$
$$= \int K_\epsilon(x,y)u(y)dy$$

where

$$K_\epsilon(x,y) = (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} a_\epsilon(x,\xi)d\xi$$
$$= (2\pi)^{-n}\widehat{a_\epsilon}(x, y-x)$$

belongs to $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n) \subset L^2(\mathbb{R}^n \times \mathbb{R}^n)$. Thus $Op_h(a_\epsilon)$ is Hilbert-Schmidt hence compact. We conclude with

$$||Op_h(a) - Op_h(a_\epsilon)||_{L^2 \to L^2} \leq C||a - a_\epsilon||_{S^{0,0},N} \to 0.$$

ΨDO (part 1): Proof of the L^2 Compactness

We let $h = 1$. Pick $a_\epsilon \in C_0^\infty$ such that $a_\epsilon \to a$ in $S^{0,0}$. Observe that

$$\begin{aligned} Op_h(a_\epsilon)u(x) &= (2\pi)^{-n} \int \int e^{i(x-y)\cdot\xi} a_\epsilon(x,\xi) u(y) dy d\xi \\ &= \int K_\epsilon(x,y) u(y) dy \end{aligned}$$

where

$$\begin{aligned} K_\epsilon(x,y) &= (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} a_\epsilon(x,\xi) d\xi \\ &= (2\pi)^{-n} \widehat{a_\epsilon}(x, y-x) \end{aligned}$$

belongs to $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n) \subset L^2(\mathbb{R}^n \times \mathbb{R}^n)$. Thus $Op_h(a_\epsilon)$ is Hilbert-Schmidt hence compact. We conclude with

$$||Op_h(a) - Op_h(a_\epsilon)||_{L^2 \to L^2} \leq C||a - a_\epsilon||_{S^{0,0},N} \to 0.$$

ΨDO (part 1): Proof of the L^2 Compactness

We let $h = 1$. Pick $a_\epsilon \in C_0^\infty$ such that $a_\epsilon \to a$ in $S^{0,0}$. Observe that

$$\begin{aligned} Op_h(a_\epsilon)u(x) &= (2\pi)^{-n} \int\int e^{i(x-y)\cdot\xi} a_\epsilon(x,\xi)u(y)dyd\xi \\ &= \int K_\epsilon(x,y)u(y)dy \end{aligned}$$

where

$$\begin{aligned} K_\epsilon(x,y) &= (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} a_\epsilon(x,\xi)d\xi \\ &= (2\pi)^{-n}\widehat{a}_\epsilon(x, y-x) \end{aligned}$$

belongs to $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n) \subset L^2(\mathbb{R}^n \times \mathbb{R}^n)$. Thus $Op_h(a_\epsilon)$ is Hilbert-Schmidt hence compact. We conclude with

$$||Op_h(a) - Op_h(a_\epsilon)||_{L^2 \to L^2} \leq C||a - a_\epsilon||_{S^{0,0},N} \to 0.$$

ΨDO (part 1): Proof of the L^2 Compactness

We let $h = 1$. Pick $a_\epsilon \in C_0^\infty$ such that $a_\epsilon \to a$ in $S^{0,0}$. Observe that

$$\begin{aligned} Op_h(a_\epsilon)u(x) &= (2\pi)^{-n} \int \int e^{i(x-y)\cdot\xi} a_\epsilon(x,\xi) u(y) dy d\xi \\ &= \int K_\epsilon(x,y) u(y) dy \end{aligned}$$

where

$$\begin{aligned} K_\epsilon(x,y) &= (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} a_\epsilon(x,\xi) d\xi \\ &= (2\pi)^{-n} \widehat{a_\epsilon}(x, y-x) \end{aligned}$$

belongs to $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n) \subset L^2(\mathbb{R}^n \times \mathbb{R}^n)$. Thus $Op_h(a_\epsilon)$ is Hilbert-Schmidt hence compact. We conclude with

$$||Op_h(a) - Op_h(a_\epsilon)||_{L^2 \to L^2} \leq C||a - a_\epsilon||_{S^{0,0},N} \to 0.$$

ΨDO (part 1): Proof of the L^2 Compactness

We let $h = 1$. Pick $a_\epsilon \in C_0^\infty$ such that $a_\epsilon \to a$ in $S^{0,0}$. Observe that

$$Op_h(a_\epsilon)u(x) = (2\pi)^{-n} \int\int e^{i(x-y)\cdot\xi} a_\epsilon(x,\xi)u(y)dyd\xi$$
$$= \int K_\epsilon(x,y)u(y)dy$$

where

$$K_\epsilon(x,y) = (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} a_\epsilon(x,\xi)d\xi$$
$$= (2\pi)^{-n}\widehat{a_\epsilon}(x, y-x)$$

belongs to $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n) \subset L^2(\mathbb{R}^n \times \mathbb{R}^n)$. Thus $Op_h(a_\epsilon)$ is Hilbert-Schmidt hence compact. We conclude with

$$||Op_h(a) - Op_h(a_\epsilon)||_{L^2 \to L^2} \leq C||a - a_\epsilon||_{S^{0,0},N} \to 0.$$

ΨDO (part 1): Proof of the L^2 Compactness

We let $h = 1$. Pick $a_\epsilon \in C_0^\infty$ such that $a_\epsilon \to a$ in $S^{0,0}$. Observe that

$$
\begin{aligned}
Op_h(a_\epsilon)u(x) &= (2\pi)^{-n} \int\int e^{i(x-y)\cdot\xi} a_\epsilon(x,\xi)u(y)dyd\xi \\
&= \int K_\epsilon(x,y)u(y)dy
\end{aligned}
$$

where

$$
\begin{aligned}
K_\epsilon(x,y) &= (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} a_\epsilon(x,\xi)d\xi \\
&= (2\pi)^{-n} \widehat{a_\epsilon}(x, y-x)
\end{aligned}
$$

belongs to $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n) \subset L^2(\mathbb{R}^n \times \mathbb{R}^n)$. Thus $Op_h(a_\epsilon)$ is Hilbert-Schmidt hence compact. We conclude with

$$
||Op_h(a) - Op_h(a_\epsilon)||_{L^2 \to L^2} \leq C||a - a_\epsilon||_{S^{0,0},N} \to 0.
$$

ΨDO (part 1): Proof of the L^2 Compactness

We let $h = 1$. Pick $a_\epsilon \in C_0^\infty$ such that $a_\epsilon \to a$ in $S^{0,0}$. Observe that

$$\begin{aligned} Op_h(a_\epsilon)u(x) &= (2\pi)^{-n} \int\int e^{i(x-y)\cdot\xi} a_\epsilon(x,\xi)u(y)dyd\xi \\ &= \int K_\epsilon(x,y)u(y)dy \end{aligned}$$

where

$$\begin{aligned} K_\epsilon(x,y) &= (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} a_\epsilon(x,\xi)d\xi \\ &= (2\pi)^{-n}\widehat{a_\epsilon}(x, y-x) \end{aligned}$$

belongs to $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n) \subset L^2(\mathbb{R}^n \times \mathbb{R}^n)$. Thus $Op_h(a_\epsilon)$ is Hilbert-Schmidt hence compact. We conclude with

$$||Op_h(a) - Op_h(a_\epsilon)||_{L^2 \to L^2} \leq C||a - a_\epsilon||_{S^{0,0},N} \to 0.$$

ΨDO (part 1): Proof of the composition formula

By density, we may work with symbols in C_0^∞. Let A_h (resp. B_h)

$$A_h(x,y) = (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} a(x, h\xi) d\xi$$

be the Schwartz kernel of $Op_h(a)$ (resp. $Op_h(b)$). The one of $Op_h(a)Op_h(b)$ is $\int A_h(x,z)B_h(z,y)dz$ i.e.

$$(2\pi)^{-2n} \int\int\int e^{i(x-z)\cdot\xi} a(x, h\xi) e^{i(z-y)\cdot\eta} b(z, h\eta) d\xi d\eta dz =$$

$$(2\pi)^{-n} \int e^{i(x-y)\cdot\eta} \underbrace{(2\pi)^{-n} \int\int e^{i(x-z)\cdot(\xi-\eta)} a(x, h\xi) b(z, h\eta) dz d\xi}_{=:c_h(x, h\eta)} d\eta$$

i.e., using $x - z = -w$ and $\xi - \eta = \zeta$,

$$c_h(x,\eta) = (2\pi)^{-n} \int\int e^{-iw\cdot\zeta} a(x, h\zeta + \eta) b(x + w, \eta) dw d\zeta$$

ΨDO (part 1): Proof of the composition formula

By density, we may work with symbols in C_0^∞. Let A_h (resp. B_h)

$$A_h(x,y) = (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} a(x, h\xi) d\xi$$

be the Schwartz kernel of $Op_h(a)$ (resp. $Op_h(b)$). The one of $Op_h(a)Op_h(b)$ is $\int A_h(x,z)B_h(z,y)dz$ i.e.

$$(2\pi)^{-2n} \int\int\int e^{i(x-z)\cdot\xi} a(x,h\xi) e^{i(z-y)\cdot\eta} b(z,h\eta) d\xi d\eta dz =$$

$$(2\pi)^{-n} \int e^{i(x-y)\cdot\eta} \underbrace{(2\pi)^{-n} \int\int e^{i(x-z)\cdot(\xi-\eta)} a(x,h\xi) b(z,h\eta) dz d\xi}_{=:c_h(x,h\eta)} d\eta$$

i.e., using $x - z = -w$ and $\xi - \eta = \zeta$,

$$c_h(x,\eta) = (2\pi)^{-n} \int\int e^{-iw\cdot\zeta} a(x, h\zeta + \eta) b(x+w, \eta) dw d\zeta$$

ΨDO (part 1): Proof of the composition formula

By density, we may work with symbols in C_0^∞. Let A_h (resp. B_h)

$$A_h(x,y) = (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} a(x,h\xi) d\xi$$

be the Schwartz kernel of $Op_h(a)$ (resp. $Op_h(b)$). The one of $Op_h(a)Op_h(b)$ is $\int A_h(x,z)B_h(z,y)dz$ i.e.

$$(2\pi)^{-2n} \int\int\int e^{i(x-z)\cdot\xi} a(x,h\xi) e^{i(z-y)\cdot\eta} b(z,h\eta) d\xi d\eta dz =$$

$$(2\pi)^{-n} \int e^{i(x-y)\cdot\eta} \underbrace{(2\pi)^{-n} \int\int e^{i(x-z)\cdot(\xi-\eta)} a(x,h\xi) b(z,h\eta) dz d\xi}_{=:c_h(x,h\eta)} d\eta$$

i.e., using $x-z=-w$ and $\xi-\eta=\zeta$,

$$c_h(x,\eta) = (2\pi)^{-n} \int\int e^{-iw\cdot\zeta} a(x,h\zeta+\eta) b(x+w,\eta) dw d\zeta$$

ΨDO (part 1): Proof of the composition formula

By density, we may work with symbols in C_0^∞. Let A_h (resp. B_h)

$$A_h(x,y) = (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} a(x, h\xi) d\xi$$

be the Schwartz kernel of $Op_h(a)$ (resp. $Op_h(b)$). The one of $Op_h(a) Op_h(b)$ is $\int A_h(x,z) B_h(z,y) dz$ i.e.

$$(2\pi)^{-2n} \int\int\int e^{i(x-z)\cdot\xi} a(x,h\xi) e^{i(z-y)\cdot\eta} b(z,h\eta) d\xi d\eta dz =$$

$$(2\pi)^{-n} \int e^{i(x-y)\cdot\eta} \underbrace{(2\pi)^{-n} \int\int e^{i(x-z)\cdot(\xi-\eta)} a(x,h\xi) b(z,h\eta) dz d\xi}_{=:c_h(x,h\eta)} d\eta$$

i.e., using $x - z = -w$ and $\xi - \eta = \zeta$,

$$c_h(x,\eta) = (2\pi)^{-n} \int\int e^{-iw\cdot\zeta} a(x, h\zeta + \eta) b(x+w, \eta) dw d\zeta$$

ΨDO (part 1): Proof of the composition formula

By density, we may work with symbols in C_0^∞. Let A_h (resp. B_h)

$$A_h(x,y) = (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} a(x, h\xi) d\xi$$

be the Schwartz kernel of $Op_h(a)$ (resp. $Op_h(b)$). The one of $Op_h(a)Op_h(b)$ is $\int A_h(x,z)B_h(z,y)dz$ i.e.

$$(2\pi)^{-2n} \int\int\int e^{i(x-z)\cdot\xi} a(x,h\xi) e^{i(z-y)\cdot\eta} b(z,h\eta) d\xi d\eta dz =$$

$$(2\pi)^{-n} \int e^{i(x-y)\cdot\eta} \underbrace{(2\pi)^{-n} \int\int e^{i(x-z)\cdot(\xi-\eta)} a(x,h\xi) b(z,h\eta) dz d\xi}_{=:c_h(x,h\eta)} d\eta$$

i.e., using $x - z = -w$ and $\xi - \eta = \zeta$,

$$c_h(x,\eta) = (2\pi)^{-n} \int\int e^{-iw\cdot\zeta} a(x, h\zeta + \eta) b(x+w, \eta) dw d\zeta$$

ΨDO (part 1): Proof of the composition formula

By density, we may work with symbols in C_0^∞. Let A_h (resp. B_h)

$$A_h(x,y) = (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} a(x, h\xi) d\xi$$

be the Schwartz kernel of $Op_h(a)$ (resp. $Op_h(b)$). The one of $Op_h(a)Op_h(b)$ is $\int A_h(x,z)B_h(z,y)dz$ i.e.

$$(2\pi)^{-2n} \int\int\int e^{i(x-z)\cdot\xi} a(x,h\xi) e^{i(z-y)\cdot\eta} b(z,h\eta) d\xi d\eta dz =$$

$$(2\pi)^{-n} \int e^{i(x-y)\cdot\eta} \underbrace{(2\pi)^{-n} \int\int e^{i(x-z)\cdot(\xi-\eta)} a(x,h\xi) b(z,h\eta) dz d\xi}_{=:c_h(x,h\eta)} d\eta$$

i.e., using $x - z = -w$ and $\xi - \eta = \zeta$,

$$c_h(x,\eta) = (2\pi)^{-n} \int\int e^{-iw\cdot\zeta} a(x, h\zeta + \eta) b(x+w, \eta) dw d\zeta$$

ΨDO (part 1): Proof of the composition formula

By density, we may work with symbols in C_0^∞. Let A_h (resp. B_h)

$$A_h(x,y) = (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} a(x,h\xi)d\xi$$

be the Schwartz kernel of $Op_h(a)$ (resp. $Op_h(b)$). The one of $Op_h(a)Op_h(b)$ is $\int A_h(x,z)B_h(z,y)dz$ i.e.

$$(2\pi)^{-2n} \int\int\int e^{i(x-z)\cdot\xi} a(x,h\xi) e^{i(z-y)\cdot\eta} b(z,h\eta) d\xi d\eta dz =$$

$$(2\pi)^{-n} \int e^{i(x-y)\cdot\eta} \underbrace{(2\pi)^{-n} \int\int e^{i(x-z)\cdot(\xi-\eta)} a(x,h\xi)b(z,h\eta)dzd\xi}_{=:c_h(x,h\eta)} d\eta$$

i.e., using $x-z=-w$ and $\xi-\eta=\zeta$,

$$c_h(x,\eta) = (2\pi)^{-n} \int\int e^{-iw\cdot\zeta} a(x,h\zeta+\eta)b(x+w,\eta)dwd\zeta$$

ΨDO (part 1): Proof of the composition formula

By density, we may work with symbols in C_0^∞. Let A_h (resp. B_h)

$$A_h(x,y) = (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} a(x, h\xi) d\xi$$

be the Schwartz kernel of $Op_h(a)$ (resp. $Op_h(b)$). The one of $Op_h(a)Op_h(b)$ is $\int A_h(x,z)B_h(z,y)dz$ i.e.

$$(2\pi)^{-2n} \int\int\int e^{i(x-z)\cdot\xi} a(x,h\xi) e^{i(z-y)\cdot\eta} b(z,h\eta) d\xi d\eta dz =$$

$$(2\pi)^{-n} \int e^{i(x-y)\cdot\eta} \underbrace{(2\pi)^{-n} \int\int e^{i(x-z)\cdot(\xi-\eta)} a(x,h\xi) b(z,h\eta) dz d\xi}_{=: c_h(x,h\eta)} d\eta$$

i.e., using $x - z = -w$ and $\xi - \eta = \zeta$,

$$c_h(x,\eta) = (2\pi)^{-n} \int\int e^{-iw\cdot\zeta} a(x, h\zeta + \eta) b(x + w, \eta) dw d\zeta$$

ΨDO (part 1): Proof of the composition formula

By density, we may work with symbols in C_0^∞. Let A_h (resp. B_h)

$$A_h(x,y) = (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} a(x,h\xi)d\xi$$

be the Schwartz kernel of $Op_h(a)$ (resp. $Op_h(b)$). The one of $Op_h(a)Op_h(b)$ is $\int A_h(x,z)B_h(z,y)dz$ i.e.

$$(2\pi)^{-2n} \int\int\int e^{i(x-z)\cdot\xi} a(x,h\xi) e^{i(z-y)\cdot\eta} b(z,h\eta)d\xi d\eta dz =$$

$$(2\pi)^{-n}\int e^{i(x-y)\cdot\eta} \underbrace{(2\pi)^{-n}\int\int e^{i(x-z)\cdot(\xi-\eta)} a(x,h\xi)b(z,h\eta)dzd\xi}_{=:c_h(x,h\eta)} d\eta$$

i.e., using $x-z=-w$ and $\xi-\eta=\zeta$,

$$c_h(x,\eta) = (2\pi)^{-n}\int\int e^{-iw\cdot\zeta} a(x,h\zeta+\eta)b(x+w,\eta)dwd\zeta$$

ΨDO (part 1): Proof of the composition formula

We integrate first in ζ

$$\begin{aligned} c_h(x,\eta) &= (2\pi)^{-n}\int\left(\int e^{-iw\cdot\zeta}a(x,h\zeta+\eta)b(x+w,\eta)d\zeta\right)dw \\ &= (2\pi h)^{-n}\int e^{i\eta\cdot w/h}\hat{a}(x,w/h)b(x+w,\eta)dw \\ &= (2\pi)^{-n}\int e^{i\eta\cdot z}\hat{a}(x,z)b(x+hz,\eta)dz \end{aligned}$$

We then Taylor expand b in hz, i.e.

$$\begin{aligned} c_h(x,\eta) &= \sum_{|\alpha|<N}\underbrace{(2\pi)^{-n}\int e^{i\eta\cdot z}\hat{a}(x,z)\frac{h^{|\alpha|}z^{\alpha}}{\alpha!}dz}_{(-ih)^{|\alpha|}(\partial_\eta^\alpha a)(x,\eta)/\alpha!}(\partial_x^\alpha b)(x,\eta) \\ &\quad + \text{Remainder (more later)} \end{aligned}$$

This yields the expansion. Is the remainder a true remainder ?

ΨDO (part 1): Proof of the composition formula

We integrate first in ζ

$$
\begin{aligned}
c_h(x,\eta) &= (2\pi)^{-n}\int\left(\int e^{-iw\cdot\zeta}a(x,h\zeta+\eta)b(x+w,\eta)d\zeta\right)dw \\
&= (2\pi h)^{-n}\int e^{i\eta\cdot w/h}\hat{a}(x,w/h)b(x+w,\eta)dw \\
&= (2\pi)^{-n}\int e^{i\eta\cdot z}\hat{a}(x,z)b(x+hz,\eta)dz
\end{aligned}
$$

We then Taylor expand b in hz, i.e.

$$
\begin{aligned}
c_h(x,\eta) &= \sum_{|\alpha|<N}\underbrace{(2\pi)^{-n}\int e^{i\eta\cdot z}\hat{a}(x,z)\frac{h^{|\alpha|}z^\alpha}{\alpha!}dz}_{(-ih)^{|\alpha|}(\partial_\eta^\alpha a)(x,\eta)/\alpha!}(\partial_x^\alpha b)(x,\eta) \\
&\quad + \text{Remainder (more later)}
\end{aligned}
$$

This yields the expansion. Is the remainder a true remainder ?

ΨDO (part 1): Proof of the composition formula

We integrate first in ζ

$$
\begin{aligned}
c_h(x,\eta) &= (2\pi)^{-n}\int\left(\int e^{-iw\cdot\zeta}a(x,h\zeta+\eta)b(x+w,\eta)d\zeta\right)dw \\
&= (2\pi h)^{-n}\int e^{i\eta\cdot w/h}\hat{a}(x,w/h)b(x+w,\eta)dw \\
&= (2\pi)^{-n}\int e^{i\eta\cdot z}\hat{a}(x,z)b(x+hz,\eta)dz
\end{aligned}
$$

We then Taylor expand b in hz, i.e.

$$
\begin{aligned}
c_h(x,\eta) &= \sum_{|\alpha|<N}\underbrace{(2\pi)^{-n}\int e^{i\eta\cdot z}\hat{a}(x,z)\frac{h^{|\alpha|}z^\alpha}{\alpha!}dz}_{(-ih)^{|\alpha|}(\partial_\eta^\alpha a)(x,\eta)/\alpha!}(\partial_x^\alpha b)(x,\eta) \\
&\quad + \text{Remainder (more later)}
\end{aligned}
$$

This yields the expansion. Is the remainder a true remainder ?

ΨDO (part 1): Proof of the composition formula

We integrate first in ζ

$$c_h(x,\eta) = (2\pi)^{-n}\int\left(\int e^{-iw\cdot\zeta}a(x,h\zeta+\eta)b(x+w,\eta)d\zeta\right)dw$$
$$= (2\pi h)^{-n}\int e^{i\eta\cdot w/h}\hat{a}(x,w/h)b(x+w,\eta)dw$$
$$= (2\pi)^{-n}\int e^{i\eta\cdot z}\hat{a}(x,z)b(x+hz,\eta)dz$$

We then Taylor expand b in hz, i.e.

$$c_h(x,\eta) = \sum_{|\alpha|<N}\underbrace{(2\pi)^{-n}\int e^{i\eta\cdot z}\hat{a}(x,z)\frac{h^{|\alpha|}z^{\alpha}}{\alpha!}dz}_{(-ih)^{|\alpha|}(\partial_\eta^\alpha a)(x,\eta)/\alpha!}(\partial_x^\alpha b)(x,\eta)$$

$+$ Remainder (more later)

This yields the expansion. Is the remainder a true remainder ?

ΨDO (part 1): Proof of the composition formula

We integrate first in ζ

$$\begin{aligned} c_h(x,\eta) &= (2\pi)^{-n}\int\left(\int e^{-iw\cdot\zeta}a(x,h\zeta+\eta)b(x+w,\eta)d\zeta\right)dw \\ &= (2\pi h)^{-n}\int e^{i\eta\cdot w/h}\hat{a}(x,w/h)b(x+w,\eta)dw \\ &= (2\pi)^{-n}\int e^{i\eta\cdot z}\hat{a}(x,z)b(x+hz,\eta)dz \end{aligned}$$

We then Taylor expand b in hz, i.e.

$$c_h(x,\eta) = \sum_{|\alpha|<N}\underbrace{(2\pi)^{-n}\int e^{i\eta\cdot z}\hat{a}(x,z)\frac{h^{|\alpha|}z^{\alpha}}{\alpha!}dz}_{(-ih)^{|\alpha|}(\partial_\eta^\alpha a)(x,\eta)/\alpha!}(\partial_x^\alpha b)(x,\eta)$$

$$+ \text{Remainder (more later)}$$

This yields the expansion. Is the remainder a true remainder ?

ΨDO (part 1): Proof of the composition formula

We integrate first in ζ

$$
\begin{aligned}
c_h(x,\eta) &= (2\pi)^{-n}\int\left(\int e^{-iw\cdot\zeta}a(x,h\zeta+\eta)b(x+w,\eta)d\zeta\right)dw \\
&= (2\pi h)^{-n}\int e^{i\eta\cdot w/h}\hat{a}(x,w/h)b(x+w,\eta)dw \\
&= (2\pi)^{-n}\int e^{i\eta\cdot z}\hat{a}(x,z)b(x+hz,\eta)dz
\end{aligned}
$$

We then Taylor expand b in hz, i.e.

$$
c_h(x,\eta) = \sum_{|\alpha|<N}\underbrace{(2\pi)^{-n}\int e^{i\eta\cdot z}\hat{a}(x,z)\frac{h^{|\alpha|}z^{\alpha}}{\alpha!}dz}_{(-ih)^{|\alpha|}(\partial_\eta^\alpha a)(x,\eta)/\alpha!}(\partial_x^\alpha b)(x,\eta)
$$

$$+ \text{Remainder (more later)}$$

This yields the expansion. Is the remainder a true remainder ?

ΨDO (part 1): Proof of the composition formula

We integrate first in ζ

$$
\begin{aligned}
c_h(x,\eta) &= (2\pi)^{-n}\int\left(\int e^{-iw\cdot\zeta}a(x,h\zeta+\eta)b(x+w,\eta)d\zeta\right)dw \\
&= (2\pi h)^{-n}\int e^{i\eta\cdot w/h}\hat{a}(x,w/h)b(x+w,\eta)dw \\
&= (2\pi)^{-n}\int e^{i\eta\cdot z}\hat{a}(x,z)b(x+hz,\eta)dz
\end{aligned}
$$

We then Taylor expand b in hz, i.e.

$$
c_h(x,\eta) = \sum_{|\alpha|<N}\underbrace{(2\pi)^{-n}\int e^{i\eta\cdot z}\hat{a}(x,z)\frac{h^{|\alpha|}z^\alpha}{\alpha!}dz}_{(-ih)^{|\alpha|}(\partial_\eta^\alpha a)(x,\eta)/\alpha!}(\partial_x^\alpha b)(x,\eta)
$$

$+$ Remainder (more later)

This yields the expansion. Is the remainder a true remainder ?

ΨDO (part 1): Proof of the composition formula

We integrate first in ζ

$$
\begin{aligned}
c_h(x,\eta) &= (2\pi)^{-n}\int\left(\int e^{-iw\cdot\zeta}a(x,h\zeta+\eta)b(x+w,\eta)d\zeta\right)dw \\
&= (2\pi h)^{-n}\int e^{i\eta\cdot w/h}\hat{a}(x,w/h)b(x+w,\eta)dw \\
&= (2\pi)^{-n}\int e^{i\eta\cdot z}\hat{a}(x,z)b(x+hz,\eta)dz
\end{aligned}
$$

We then Taylor expand b in hz, i.e.

$$
c_h(x,\eta) = \sum_{|\alpha|<N}\underbrace{(2\pi)^{-n}\int e^{i\eta\cdot z}\hat{a}(x,z)\frac{h^{|\alpha|}z^{\alpha}}{\alpha!}dz}_{(-ih)^{|\alpha|}(\partial_\eta^\alpha a)(x,\eta)/\alpha!}(\partial_x^\alpha b)(x,\eta)
$$

$+$ Remainder (more later)

This yields the expansion. Is the remainder a true remainder ?

ΨDO (part 1): Proof of the composition formula

We integrate first in ζ

$$\begin{aligned} c_h(x,\eta) &= (2\pi)^{-n}\int\left(\int e^{-iw\cdot\zeta}a(x,h\zeta+\eta)b(x+w,\eta)d\zeta\right)dw \\ &= (2\pi h)^{-n}\int e^{i\eta\cdot w/h}\hat{a}(x,w/h)b(x+w,\eta)dw \\ &= (2\pi)^{-n}\int e^{i\eta\cdot z}\hat{a}(x,z)b(x+hz,\eta)dz \end{aligned}$$

We then Taylor expand b in hz, i.e.

$$c_h(x,\eta) = \sum_{|\alpha|<N}\underbrace{(2\pi)^{-n}\int e^{i\eta\cdot z}\hat{a}(x,z)\frac{h^{|\alpha|}z^\alpha}{\alpha!}dz}_{(-ih)^{|\alpha|}(\partial_\eta^\alpha a)(x,\eta)/\alpha!}(\partial_x^\alpha b)(x,\eta)$$

$$+ \text{Remainder (more later)}$$

This yields the expansion. Is the remainder a true remainder ?

ΨDO (part 1): Proof of the composition formula

We integrate first in ζ

$$
\begin{aligned}
c_h(x,\eta) &= (2\pi)^{-n}\int\left(\int e^{-iw\cdot\zeta}a(x,h\zeta+\eta)b(x+w,\eta)d\zeta\right)dw \\
&= (2\pi h)^{-n}\int e^{i\eta\cdot w/h}\hat{a}(x,w/h)b(x+w,\eta)dw \\
&= (2\pi)^{-n}\int e^{i\eta\cdot z}\hat{a}(x,z)b(x+hz,\eta)dz
\end{aligned}
$$

We then Taylor expand b in hz, i.e.

$$
c_h(x,\eta) = \sum_{|\alpha|<N}\underbrace{(2\pi)^{-n}\int e^{i\eta\cdot z}\hat{a}(x,z)\frac{h^{|\alpha|}z^\alpha}{\alpha!}dz}_{(-ih)^{|\alpha|}(\partial_\eta^\alpha a)(x,\eta)/\alpha!}(\partial_x^\alpha b)(x,\eta)
$$

$$+ \text{Remainder (more later)}$$

This yields the expansion. Is the remainder a true remainder ?

ΨDO (part 1): Proof of the composition formula

We integrate first in ζ

$$\begin{aligned} c_h(x,\eta) &= (2\pi)^{-n}\int\left(\int e^{-iw\cdot\zeta}a(x,h\zeta+\eta)b(x+w,\eta)d\zeta\right)dw \\ &= (2\pi h)^{-n}\int e^{i\eta\cdot w/h}\hat{a}(x,w/h)b(x+w,\eta)dw \\ &= (2\pi)^{-n}\int e^{i\eta\cdot z}\hat{a}(x,z)b(x+hz,\eta)dz \end{aligned}$$

We then Taylor expand b in hz, i.e.

$$c_h(x,\eta) = \sum_{|\alpha|<N}\underbrace{(2\pi)^{-n}\int e^{i\eta\cdot z}\hat{a}(x,z)\frac{h^{|\alpha|}z^{\alpha}}{\alpha!}dz}_{(-ih)^{|\alpha|}(\partial_\eta^\alpha a)(x,\eta)/\alpha!}(\partial_x^\alpha b)(x,\eta)$$

$$+ \text{Remainder (more later)}$$

This yields the expansion. Is the remainder a true remainder ?

ΨDO (part 1): Proof of the composition formula

We integrate first in ζ

$$\begin{aligned} c_h(x,\eta) &= (2\pi)^{-n} \int \left(\int e^{-iw\cdot\zeta} a(x, h\zeta+\eta) b(x+w,\eta) d\zeta \right) dw \\ &= (2\pi h)^{-n} \int e^{i\eta\cdot w/h} \hat{a}(x, w/h) b(x+w,\eta) dw \\ &= (2\pi)^{-n} \int e^{i\eta\cdot z} \hat{a}(x,z) b(x+hz,\eta) dz \end{aligned}$$

We then Taylor expand b in hz, i.e.

$$c_h(x,\eta) = \sum_{|\alpha|<N} \underbrace{(2\pi)^{-n} \int e^{i\eta\cdot z} \hat{a}(x,z) \frac{h^{|\alpha|} z^\alpha}{\alpha!} dz}_{(-ih)^{|\alpha|}(\partial_\eta^\alpha a)(x,\eta)/\alpha!} (\partial_x^\alpha b)(x,\eta)$$

$$+ \text{ Remainder (more later)}$$

This yields the expansion. Is the remainder a true remainder ?

ΨDO (part 1): Proof of the composition formula

The remainder terms associated to the previous expression are of the form, with $|\alpha| = N$

$$(2\pi)^{-n} \int_0^1 (1-t)^{N-1} \int e^{i\eta\cdot z} \hat{a}(x,z)(\partial_x^\alpha b)(x+thz,\eta)\frac{(hz)^\alpha}{\alpha!} dz dt$$

Dropping the integration in t, we get for each $t > 0$

$$h^N \int e^{i\eta\cdot z} \underbrace{z^\alpha \hat{a}(x,z)}_{\widehat{D_\xi^\alpha a}(x,z)} (\partial_x^\alpha b)(x+thz,\eta) dz =$$

$$h^N \int\int e^{i(\eta-\xi)\cdot z} D_\xi^\alpha a(x,\xi)(\partial_x^\alpha b)(x+thz,\eta) d\xi dz$$

ΨDO (part 1): Proof of the composition formula

The remainder terms associated to the previous expression are of the form, with $|\alpha| = N$

$$(2\pi)^{-n} \int_0^1 (1-t)^{N-1} \int e^{i\eta\cdot z} \hat{a}(x,z)(\partial_x^\alpha b)(x+thz,\eta)\frac{(hz)^\alpha}{\alpha!} dzdt$$

Dropping the integration in t, we get for each $t > 0$

$$h^N \int e^{i\eta\cdot z} \underbrace{z^\alpha \hat{a}(x,z)}_{\widehat{D_\xi^\alpha a}(x,z)}(\partial_x^\alpha b)(x+thz,\eta)dz =$$

$$h^N \int\int e^{i(\eta-\xi)\cdot z} D_\xi^\alpha a(x,\xi)(\partial_x^\alpha b)(x+thz,\eta)d\xi dz$$

ΨDO (part 1): Proof of the composition formula

The remainder terms associated to the previous expression are of the form, with $|\alpha| = N$

$$(2\pi)^{-n} \int_0^1 (1-t)^{N-1} \int e^{i\eta \cdot z} \hat{a}(x,z) (\partial_x^\alpha b)(x+thz,\eta) \frac{(hz)^\alpha}{\alpha!} dz dt$$

Dropping the integration in t, we get for each $t > 0$

$$h^N \int e^{i\eta \cdot z} \underbrace{z^\alpha \hat{a}(x,z)}_{\widehat{D_\xi^\alpha a}(x,z)} (\partial_x^\alpha b)(x+thz,\eta) dz =$$

$$h^N \int \int e^{i(\eta-\xi)\cdot z} D_\xi^\alpha a(x,\xi) (\partial_x^\alpha b)(x+thz,\eta) d\xi dz$$

ΨDO (part 1): Proof of the composition formula

The remainder terms associated to the previous expression are of the form, with $|\alpha| = N$

$$(2\pi)^{-n}\int_0^1 (1-t)^{N-1}\int e^{i\eta\cdot z}\hat{a}(x,z)(\partial_x^\alpha b)(x+thz,\eta)\frac{(hz)^\alpha}{\alpha!}dzdt$$

Dropping the integration in t, we get for each $t > 0$

$$h^N \int e^{i\eta\cdot z}\underbrace{z^\alpha \hat{a}(x,z)}_{\widehat{D_\xi^\alpha a}(x,z)}(\partial_x^\alpha b)(x+thz,\eta)dz =$$

$$h^N \int\int e^{i(\eta-\xi)\cdot z} D_\xi^\alpha a(x,\xi)(\partial_x^\alpha b)(x+thz,\eta)d\xi dz$$

ΨDO (part 1): Proof of the composition formula

The remainder terms associated to the previous expression are of the form, with $|\alpha| = N$

$$(2\pi)^{-n} \int_0^1 (1-t)^{N-1} \int e^{i\eta\cdot z} \hat{a}(x,z)(\partial_x^\alpha b)(x+thz,\eta) \frac{(hz)^\alpha}{\alpha!} dzdt$$

Dropping the integration in t, we get for each $t > 0$

$$h^N \int e^{i\eta\cdot z} \underbrace{z^\alpha \hat{a}(x,z)}_{\widehat{D_\xi^\alpha a}(x,z)} (\partial_x^\alpha b)(x+thz,\eta) dz =$$

$$h^N \int\int e^{i(\eta-\xi)\cdot z} D_\xi^\alpha a(x,\xi)(\partial_x^\alpha b)(x+thz,\eta) d\xi dz$$

ΨDO (part 1): Proof of the composition formula

The remainder terms associated to the previous expression are of the form, with $|\alpha| = N$

$$(2\pi)^{-n}\int_0^1 (1-t)^{N-1}\int e^{i\eta\cdot z}\hat{a}(x,z)(\partial_x^\alpha b)(x+thz,\eta)\frac{(hz)^\alpha}{\alpha!}dzdt$$

Dropping the integration in t, we get for each $t > 0$

$$h^N\int e^{i\eta\cdot z}\underbrace{z^\alpha\hat{a}(x,z)}_{\widehat{D_\xi^\alpha a}(x,z)}(\partial_x^\alpha b)(x+thz,\eta)dz =$$

$$h^N\int\int e^{i(\eta-\xi)\cdot z}D_\xi^\alpha a(x,\xi)(\partial_x^\alpha b)(x+thz,\eta)d\xi dz$$

ΨDO (part 1): Proof of the composition formula

To complete the proof, we wish to gain additional polynomial decay in z and ξ by integrating by part using

$$\frac{1}{(|z|^2+1)^M}(1-\Delta_\xi)^M e^{i(\eta-\xi)\cdot z} = e^{i(\eta-\xi)\cdot z}$$

and

$$\frac{1}{(|\eta-\xi|^2+1)^M}(1-\Delta_z)^M e^{i(\eta-\xi)\cdot z} = e^{i(\eta-\xi)\cdot z}$$

so we end up with integrals of the form (if we view a and b as symbols in $S^{\mu,m}$ and $S^{\mu',m'}$, as will eventually be the case)

$$\int\int O(\langle z\rangle^{-2M})O(\langle \xi-\eta\rangle^{-2M})O(\langle x\rangle^{\mu}\langle \xi\rangle^{m-N})O(\langle x+thz\rangle^{\mu'}\langle \eta\rangle^{m'})d\xi dz$$

i.e.

$$\int\int O(\langle z\rangle^{-2M})O(\langle \xi\rangle^{-2M})O(\langle x\rangle^{\mu}\langle \xi+\eta\rangle^{m-N})O(\langle x+thz\rangle^{\mu'}\langle \eta\rangle^{m'})d\xi dz$$

ΨDO (part 1): Proof of the composition formula

To complete the proof, we wish to gain additional polynomial decay in z and ξ by integrating by part using

$$\frac{1}{(|z|^2+1)^M}(1-\Delta_\xi)^M e^{i(\eta-\xi)\cdot z} = e^{i(\eta-\xi)\cdot z}$$

and

$$\frac{1}{(|\eta-\xi|^2+1)^M}(1-\Delta_z)^M e^{i(\eta-\xi)\cdot z} = e^{i(\eta-\xi)\cdot z}$$

so we end up with integrals of the form (if we view a and b as symbols in $S^{\mu,m}$ and $S^{\mu',m'}$, as will eventually be the case)

$$\iint O(\langle z\rangle^{-2M})O(\langle \xi-\eta\rangle^{-2M})O(\langle x\rangle^{\mu}\langle \xi\rangle^{m-N})O(\langle x+thz\rangle^{\mu'}\langle \eta\rangle^{m'})d\xi dz$$

i.e.

$$\iint O(\langle z\rangle^{-2M})O(\langle \xi\rangle^{-2M})O(\langle x\rangle^{\mu}\langle \xi+\eta\rangle^{m-N})O(\langle x+thz\rangle^{\mu'}\langle \eta\rangle^{m'})d\xi dz$$

ΨDO (part 1): Proof of the composition formula

To complete the proof, we wish to gain additional polynomial decay in z and ξ by integrating by part using

$$\frac{1}{(|z|^2+1)^M}(1-\Delta_\xi)^M e^{i(\eta-\xi)\cdot z} = e^{i(\eta-\xi)\cdot z}$$

and

$$\frac{1}{(|\eta-\xi|^2+1)^M}(1-\Delta_z)^M e^{i(\eta-\xi)\cdot z} = e^{i(\eta-\xi)\cdot z}$$

so we end up with integrals of the form (if we view a and b as symbols in $S^{\mu,m}$ and $S^{\mu',m'}$, as will eventually be the case)

$$\int\int O(\langle z\rangle^{-2M})O(\langle\xi-\eta\rangle^{-2M})O(\langle x\rangle^{\mu}\langle\xi\rangle^{m-N})O(\langle x+thz\rangle^{\mu'}\langle\eta\rangle^{m'})d\xi dz$$

i.e.

$$\int\int O(\langle z\rangle^{-2M})O(\langle\xi\rangle^{-2M})O(\langle x\rangle^{\mu}\langle\xi+\eta\rangle^{m-N})O(\langle x+thz\rangle^{\mu'}\langle\eta\rangle^{m'})d\xi dz$$

ΨDO (part 1): Proof of the composition formula

To complete the proof, we wish to gain additional polynomial decay in z and ξ by integrating by part using

$$\frac{1}{(|z|^2+1)^M}(1-\Delta_\xi)^M e^{i(\eta-\xi)\cdot z} = e^{i(\eta-\xi)\cdot z}$$

and

$$\frac{1}{(|\eta-\xi|^2+1)^M}(1-\Delta_z)^M e^{i(\eta-\xi)\cdot z} = e^{i(\eta-\xi)\cdot z}$$

so we end up with integrals of the form (if we view a and b as symbols in $S^{\mu,m}$ and $S^{\mu',m'}$, as will eventually be the case)

$$\iint O(\langle z\rangle^{-2M})O(\langle \xi-\eta\rangle^{-2M})O(\langle x\rangle^{\mu}\langle \xi\rangle^{m-N})O(\langle x+thz\rangle^{\mu'}\langle \eta\rangle^{m'})d\xi dz$$

i.e.

$$\iint O(\langle z\rangle^{-2M})O(\langle \xi\rangle^{-2M})O(\langle x\rangle^{\mu}\langle \xi+\eta\rangle^{m-N})O(\langle x+thz\rangle^{\mu'}\langle \eta\rangle^{m'})d\xi dz$$

ΨDO (part 1): Proof of the composition formula

To complete the proof, we wish to gain additional polynomial decay in z and ξ by integrating by part using

$$\frac{1}{(|z|^2+1)^M}(1-\Delta_\xi)^M e^{i(\eta-\xi)\cdot z} = e^{i(\eta-\xi)\cdot z}$$

and

$$\frac{1}{(|\eta-\xi|^2+1)^M}(1-\Delta_z)^M e^{i(\eta-\xi)\cdot z} = e^{i(\eta-\xi)\cdot z}$$

so we end up with integrals of the form (if we view a and b as symbols in $S^{\mu,m}$ and $S^{\mu',m'}$, as will eventually be the case)

$$\iint O(\langle z\rangle^{-2M})O(\langle \xi-\eta\rangle^{-2M})O(\langle x\rangle^{\mu}\langle \xi\rangle^{m-N})O(\langle x+thz\rangle^{\mu'}\langle \eta\rangle^{m'})d\xi dz$$

i.e.

$$\iint O(\langle z\rangle^{-2M})O(\langle \xi\rangle^{-2M})O(\langle x\rangle^{\mu}\langle \xi+\eta\rangle^{m-N})O(\langle x+thz\rangle^{\mu'}\langle \eta\rangle^{m'})d\xi dz$$

ΨDO (part 1): Proof of the composition formula

To complete the proof, we wish to gain additional polynomial decay in z and ξ by integrating by part using

$$\frac{1}{(|z|^2+1)^M}(1-\Delta_\xi)^M e^{i(\eta-\xi)\cdot z} = e^{i(\eta-\xi)\cdot z}$$

and

$$\frac{1}{(|\eta-\xi|^2+1)^M}(1-\Delta_z)^M e^{i(\eta-\xi)\cdot z} = e^{i(\eta-\xi)\cdot z}$$

so we end up with integrals of the form (if we view a and b as symbols in $S^{\mu,m}$ and $S^{\mu',m'}$, as will eventually be the case)

$$\iint O(\langle z\rangle^{-2M})O(\langle \xi-\eta\rangle^{-2M})O(\langle x\rangle^{\mu}\langle \xi\rangle^{m-N})O(\langle x+thz\rangle^{\mu'}\langle \eta\rangle^{m'})d\xi dz$$

i.e.

$$\iint O(\langle z\rangle^{-2M})O(\langle \xi\rangle^{-2M})O(\langle x\rangle^{\mu}\langle \xi+\eta\rangle^{m-N})O(\langle x+thz\rangle^{\mu'}\langle \eta\rangle^{m'})d\xi dz$$

ΨDO (part 1): Proof of the composition formula (end)

Using the Peetre inequality

$$\langle x + thz\rangle^{\mu'} \lesssim \langle x\rangle^{\mu'} \underbrace{\langle thz\rangle^{|\mu'|}}_{\text{polynomial loss}}$$

$$\langle \xi + \eta\rangle^{m-N} \lesssim \langle \eta\rangle^{m-N} \underbrace{\langle \xi\rangle^{|m-N|}}_{\text{polynomial loss}}$$

the terms

$$\iint O(\langle z\rangle^{-2M})O(\langle \xi\rangle^{-2M})O(\langle x\rangle^{\mu}\langle \xi+\eta\rangle^{m-N})O(\langle x+thz\rangle^{\mu'}\langle \eta\rangle^{m'})d\xi dz$$

are bounded by

$$\langle x\rangle^{\mu+\mu'}\langle \eta\rangle^{m+m'-N}\iint \underbrace{O(\langle z\rangle^{-2M+|\mu'|})O(\langle \xi\rangle^{-2M+|m-N|})}_{\text{integrable if } M\gg 1} d\xi dz \qquad \square$$

ΨDO (part 1): Proof of the composition formula (end)

Using the Peetre inequality

$$\langle x + thz\rangle^{\mu'} \lesssim \langle x\rangle^{\mu'} \underbrace{\langle thz\rangle^{|\mu'|}}_{\text{polynomial loss}}$$

$$\langle \xi + \eta\rangle^{m-N} \lesssim \langle \eta\rangle^{m-N} \underbrace{\langle \xi\rangle^{|m-N|}}_{\text{polynomial loss}}$$

the terms

$$\iint O(\langle z\rangle^{-2M})O(\langle \xi\rangle^{-2M})O(\langle x\rangle^{\mu}\langle \xi+\eta\rangle^{m-N})O(\langle x+thz\rangle^{\mu'}\langle \eta\rangle^{m'})d\xi dz$$

are bounded by

$$\langle x\rangle^{\mu+\mu'}\langle \eta\rangle^{m+m'-N}\iint \underbrace{O(\langle z\rangle^{-2M+|\mu'|})O(\langle \xi\rangle^{-2M+|m-N|})}_{\text{integrable if } M\gg 1} d\xi dz \qquad \square$$

ΨDO (part 1): Proof of the composition formula (end)

Using the Peetre inequality

$$\langle x + thz\rangle^{\mu'} \lesssim \langle x\rangle^{\mu'} \underbrace{\langle thz\rangle^{|\mu'|}}_{\text{polynomial loss}}$$

$$\langle \xi + \eta\rangle^{m-N} \lesssim \langle \eta\rangle^{m-N} \underbrace{\langle \xi\rangle^{|m-N|}}_{\text{polynomial loss}}$$

the terms

$$\iint O(\langle z\rangle^{-2M})O(\langle \xi\rangle^{-2M})O(\langle x\rangle^{\mu}\langle \xi+\eta\rangle^{m-N})O(\langle x+thz\rangle^{\mu'}\langle \eta\rangle^{m'})d\xi dz$$

are bounded by

$$\langle x\rangle^{\mu+\mu'}\langle \eta\rangle^{m+m'-N}\iint \underbrace{O(\langle z\rangle^{-2M+|\mu'|})O(\langle \xi\rangle^{-2M+|m-N|})}_{\text{integrable if } M\gg 1} d\xi dz \qquad \square$$

ΨDO (part 1): Proof of the composition formula (end)

Using the Peetre inequality

$$\langle x + thz\rangle^{\mu'} \lesssim \langle x\rangle^{\mu'} \underbrace{\langle thz\rangle^{|\mu'|}}_{\text{polynomial loss}}$$

$$\langle \xi + \eta\rangle^{m-N} \lesssim \langle \eta\rangle^{m-N} \underbrace{\langle \xi\rangle^{|m-N|}}_{\text{polynomial loss}}$$

the terms

$$\iint O(\langle z\rangle^{-2M}) O(\langle \xi\rangle^{-2M}) O(\langle x\rangle^{\mu} \langle \xi + \eta\rangle^{m-N}) O(\langle x + thz\rangle^{\mu'} \langle \eta\rangle^{m'}) d\xi dz$$

are bounded by

$$\langle x\rangle^{\mu+\mu'} \langle \eta\rangle^{m+m'-N} \iint \underbrace{O(\langle z\rangle^{-2M+|\mu'|}) O(\langle \xi\rangle^{-2M+|m-N|})}_{\text{integrable if } M \gg 1} d\xi dz \qquad \square$$

ΨDO (part 1): Proof of the composition formula (end)

Using the Peetre inequality

$$\langle x + thz\rangle^{\mu'} \lesssim \langle x\rangle^{\mu'} \underbrace{\langle thz\rangle^{|\mu'|}}_{\text{polynomial loss}}$$

$$\langle \xi + \eta\rangle^{m-N} \lesssim \langle \eta\rangle^{m-N} \underbrace{\langle \xi\rangle^{|m-N|}}_{\text{polynomial loss}}$$

the terms

$$\iint O(\langle z\rangle^{-2M}) O(\langle \xi\rangle^{-2M}) O(\langle x\rangle^{\mu}\langle \xi+\eta\rangle^{m-N}) O(\langle x+thz\rangle^{\mu'}\langle \eta\rangle^{m'}) d\xi dz$$

are bounded by

$$\langle x\rangle^{\mu+\mu'}\langle \eta\rangle^{m+m'-N} \iint \underbrace{O(\langle z\rangle^{-2M+|\mu'|}) O(\langle \xi\rangle^{-2M+|m-N|})}_{\text{integrable if } M\gg 1} d\xi dz \qquad \square$$

ΨDO (part 1): Proof of the composition formula (end)

Using the Peetre inequality

$$\langle x + thz\rangle^{\mu'} \lesssim \langle x\rangle^{\mu'} \underbrace{\langle thz\rangle^{|\mu'|}}_{\text{polynomial loss}}$$

$$\langle \xi + \eta\rangle^{m-N} \lesssim \langle \eta\rangle^{m-N} \underbrace{\langle \xi\rangle^{|m-N|}}_{\text{polynomial loss}}$$

the terms

$$\iint O(\langle z\rangle^{-2M})O(\langle \xi\rangle^{-2M})O(\langle x\rangle^{\mu}\langle \xi+\eta\rangle^{m-N})O(\langle x+thz\rangle^{\mu'}\langle \eta\rangle^{m'})d\xi dz$$

are bounded by

$$\langle x\rangle^{\mu+\mu'}\langle \eta\rangle^{m+m'-N} \iint \underbrace{O(\langle z\rangle^{-2M+|\mu'|})O(\langle \xi\rangle^{-2M+|m-N|})}_{\text{integrable if } M \gg 1} d\xi dz \qquad \square$$

ΨDO (part 1): Proof of the composition formula (end)

Using the Peetre inequality

$$\langle x + thz\rangle^{\mu'} \lesssim \langle x\rangle^{\mu'} \underbrace{\langle thz\rangle^{|\mu'|}}_{\text{polynomial loss}}$$

$$\langle \xi + \eta\rangle^{m-N} \lesssim \langle \eta\rangle^{m-N} \underbrace{\langle \xi\rangle^{|m-N|}}_{\text{polynomial loss}}$$

the terms

$$\iint O(\langle z\rangle^{-2M})O(\langle \xi\rangle^{-2M})O(\langle x\rangle^{\mu}\langle \xi+\eta\rangle^{m-N})O(\langle x+thz\rangle^{\mu'}\langle \eta\rangle^{m'})d\xi dz$$

are bounded by

$$\langle x\rangle^{\mu+\mu'}\langle \eta\rangle^{m+m'-N}\iint \underbrace{O(\langle z\rangle^{-2M+|\mu'|})O(\langle \xi\rangle^{-2M+|m-N|})}_{\text{integrable if } M\gg 1} d\xi dz \qquad \square$$

ΨDO (part 1): Proof of the composition formula (end)

Using the Peetre inequality

$$\langle x + thz\rangle^{\mu'} \lesssim \langle x\rangle^{\mu'} \underbrace{\langle thz\rangle^{|\mu'|}}_{\text{polynomial loss}}$$

$$\langle \xi + \eta\rangle^{m-N} \lesssim \langle \eta\rangle^{m-N} \underbrace{\langle \xi\rangle^{|m-N|}}_{\text{polynomial loss}}$$

the terms

$$\int\int O(\langle z\rangle^{-2M})O(\langle \xi\rangle^{-2M})O(\langle x\rangle^{\mu}\langle \xi+\eta\rangle^{m-N})O(\langle x+thz\rangle^{\mu'}\langle \eta\rangle^{m'})d\xi dz$$

are bounded by

$$\langle x\rangle^{\mu+\mu'}\langle \eta\rangle^{m+m'-N}\int\int \underbrace{O(\langle z\rangle^{-2M+|\mu'|})O(\langle \xi\rangle^{-2M+|m-N|})}_{\text{integrable if } M\gg 1} d\xi dz \qquad \square$$

ΨDO (part 1): Proof of the composition formula (end)

Using the Peetre inequality

$$\langle x+thz\rangle^{\mu'} \lesssim \langle x\rangle^{\mu'} \underbrace{\langle thz\rangle^{|\mu'|}}_{\text{polynomial loss}}$$

$$\langle \xi+\eta\rangle^{m-N} \lesssim \langle \eta\rangle^{m-N} \underbrace{\langle \xi\rangle^{|m-N|}}_{\text{polynomial loss}}$$

the terms

$$\iint O(\langle z\rangle^{-2M})O(\langle \xi\rangle^{-2M})O(\langle x\rangle^{\mu}\langle \xi+\eta\rangle^{m-N})O(\langle x+thz\rangle^{\mu'}\langle \eta\rangle^{m'})d\xi dz$$

are bounded by

$$\langle x\rangle^{\mu+\mu'}\langle \eta\rangle^{m+m'-N}\iint \underbrace{O(\langle z\rangle^{-2M+|\mu'|})O(\langle \xi\rangle^{-2M+|m-N|})}_{\text{integrable if } M\gg 1} d\xi dz \qquad \square$$

ΨDO (part 1): Proof of the composition formula (end)

Using the Peetre inequality

$$\langle x + thz\rangle^{\mu'} \lesssim \langle x\rangle^{\mu'} \underbrace{\langle thz\rangle^{|\mu'|}}_{\text{polynomial loss}}$$

$$\langle \xi + \eta\rangle^{m-N} \lesssim \langle \eta\rangle^{m-N} \underbrace{\langle \xi\rangle^{|m-N|}}_{\text{polynomial loss}}$$

the terms

$$\iint O(\langle z\rangle^{-2M}) O(\langle \xi\rangle^{-2M}) O(\langle x\rangle^{\mu} \langle \xi + \eta\rangle^{m-N}) O(\langle x + thz\rangle^{\mu'} \langle \eta\rangle^{m'}) d\xi dz$$

are bounded by

$$\langle x\rangle^{\mu+\mu'} \langle \eta\rangle^{m+m'-N} \iint \underbrace{O(\langle z\rangle^{-2M+|\mu'|}) O(\langle \xi\rangle^{-2M+|m-N|})}_{\text{integrable if } M \gg 1} d\xi dz \qquad \square$$

ΨDO (part 2): More general classes

Definition. A positive function m on $\mathbb{R}^{2n}$ is an order function is there exist C and M such that

$$m(x,\xi) \leq Cm(y,\eta)(1+|x-y|+|\xi-\eta|)^M.$$

Examples. $m(x,\xi) = \langle x\rangle^\mu \langle\xi\rangle^\nu$
In $\mathbb{R}^3 \times \mathbb{R}^3$

$$m(v,\eta) = \langle v\rangle^\gamma (1+|\eta|^2+|\eta\wedge v|^2+|v^2|)^s$$

with $0 < s < 1$ and $\gamma > -3$ (related to the linearized Boltzmann operator)

Definition. Let $g_0 = dx^2 + d\xi^2$ the standard Euclidean metric on $\mathbb{R}^{2n}$. If m is an order function, $S(m, g_0)$ is the space of smooth functions a such that

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq C_{\alpha\beta} m(x,\xi).$$

ΨDO (part 2): More general classes

Definition. A positive function m on $\mathbb{R}^{2n}$ is an order function is there exist C and M such that

$$m(x,\xi) \leq Cm(y,\eta)(1+|x-y|+|\xi-\eta|)^M.$$

Examples. $m(x,\xi) = \langle x\rangle^\mu \langle\xi\rangle^\nu$
In $\mathbb{R}^3 \times \mathbb{R}^3$

$$m(v,\eta) = \langle v\rangle^\gamma (1+|\eta|^2+|\eta\wedge v|^2+|v^2|)^s$$

with $0 < s < 1$ and $\gamma > -3$ (related to the linearized Boltzmann operator)

Definition. Let $g_0 = dx^2 + d\xi^2$ the standard Euclidean metric on $\mathbb{R}^{2n}$. If m is an order function, $S(m,g_0)$ is the space of smooth functions a such that

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq C_{\alpha\beta} m(x,\xi).$$

ΨDO (part 2): More general classes

Definition. A positive function m on $\mathbb{R}^{2n}$ is an order function is there exist C and M such that

$$m(x,\xi) \leq Cm(y,\eta)(1+|x-y|+|\xi-\eta|)^M.$$

Examples. $m(x,\xi) = \langle x\rangle^\mu \langle \xi\rangle^\nu$

In $\mathbb{R}^3 \times \mathbb{R}^3$

$$m(v,\eta) = \langle v\rangle^\gamma (1+|\eta|^2+|\eta\wedge v|^2+|v^2|)^s$$

with $0 < s < 1$ and $\gamma > -3$ (related to the linearized Boltzmann operator)

Definition. Let $g_0 = dx^2 + d\xi^2$ the standard Euclidean metric on $\mathbb{R}^{2n}$. If m is an order function, $S(m, g_0)$ is the space of smooth functions a such that

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq C_{\alpha\beta} m(x,\xi).$$

ΨDO (part 2): More general classes

Definition. A positive function m on $\mathbb{R}^{2n}$ is an order function is there exist C and M such that

$$m(x,\xi) \leq Cm(y,\eta)(1+|x-y|+|\xi-\eta|)^M.$$

Examples. $m(x,\xi) = \langle x\rangle^\mu \langle\xi\rangle^\nu$
In $\mathbb{R}^3 \times \mathbb{R}^3$

$$m(v,\eta) = \langle v\rangle^\gamma (1+|\eta|^2+|\eta\wedge v|^2+|v^2|)^s$$

with $0 < s < 1$ and $\gamma > -3$ (related to the linearized Boltzmann operator)

Definition. Let $g_0 = dx^2 + d\xi^2$ the standard Euclidean metric on $\mathbb{R}^{2n}$. If m is an order function, $S(m,g_0)$ is the space of smooth functions a such that

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq C_{\alpha\beta} m(x,\xi).$$

ΨDO (part 2): More general classes

Definition. A positive function m on $\mathbb{R}^{2n}$ is an order function is there exist C and M such that

$$m(x,\xi) \leq Cm(y,\eta)(1+|x-y|+|\xi-\eta|)^M.$$

Examples. $m(x,\xi) = \langle x\rangle^\mu \langle\xi\rangle^\nu$
In $\mathbb{R}^3 \times \mathbb{R}^3$

$$m(v,\eta) = \langle v\rangle^\gamma (1+|\eta|^2+|\eta\wedge v|^2+|v^2|)^s$$

with $0 < s < 1$ and $\gamma > -3$ (related to the linearized Boltzmann operator)

Definition. Let $g_0 = dx^2 + d\xi^2$ the standard Euclidean metric on $\mathbb{R}^{2n}$. If m is an order function, $S(m, g_0)$ is the space of smooth functions a such that

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq C_{\alpha\beta} m(x,\xi).$$

ΨDO (part 2): More general classes

Definition. A positive function m on $\mathbb{R}^{2n}$ is an order function is there exist C and M such that

$$m(x,\xi) \leq Cm(y,\eta)(1+|x-y|+|\xi-\eta|)^M.$$

Examples. $m(x,\xi) = \langle x\rangle^\mu \langle \xi\rangle^\nu$
In $\mathbb{R}^3 \times \mathbb{R}^3$

$$m(v,\eta) = \langle v\rangle^\gamma (1+|\eta|^2+|\eta\wedge v|^2+|v^2|)^s$$

with $0 < s < 1$ and $\gamma > -3$ (related to the linearized Boltzmann operator)

Definition. Let $g_0 = dx^2 + d\xi^2$ the standard Euclidean metric on $\mathbb{R}^{2n}$. If m is an order function, $S(m, g_0)$ is the space of smooth functions a such that

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq C_{\alpha\beta} m(x,\xi).$$

ΨDO (part 2): More general classes

There is a general notion of Riemannian metrics for pseudo-differential calculus (Hörmander). We just give a few examples

$$g_{\mathrm{class}} = dx^2 + \frac{d\xi^2}{\langle \xi \rangle^2}, \quad g_{\mathrm{scat}} = \frac{dx^2}{\langle x \rangle^2} + \frac{d\xi^2}{\langle \xi \rangle^2}, \quad g_{\rho,\delta} = \frac{dx^2}{\langle \xi \rangle^{-2\delta}} + \frac{d\xi^2}{\langle \xi \rangle^{2\rho}}$$

($0 \le \delta \le \rho \le 1$ and $\delta < 1$) and some related symbol classes

$$a \in S(\langle \xi \rangle^m, g_{\mathrm{class}}) = S^m \quad \Leftrightarrow \quad \partial_x^\alpha \partial_\xi^\beta a(x,\xi) = O(\langle \xi \rangle^{m-|\beta|})$$

$$a \in S(\langle x \rangle^\mu \langle \xi \rangle^m, g_{\mathrm{scat}}) \quad \Leftrightarrow \quad \partial_x^\alpha \partial_\xi^\beta a(x,\xi) = O(\langle x \rangle^{\mu-|\alpha|} \langle \xi \rangle^{m-|\beta|})$$

$$a \in S(\langle \xi \rangle^m, g_{\rho,\delta}) =: S^m_{\rho,\delta} \quad \Leftrightarrow \quad \partial_x^\alpha \partial_\xi^\beta a(x,\xi) = O(\langle \xi \rangle^{m+\delta|\alpha|-\rho|\beta|})$$

ΨDO (part 2): More general classes

There is a general notion of Riemannian metrics for pseudo-differential calculus (Hörmander). We just give a few examples

$$g_{\text{class}} = dx^2 + \frac{d\xi^2}{\langle\xi\rangle^2}, \quad g_{\text{scat}} = \frac{dx^2}{\langle x\rangle^2} + \frac{d\xi^2}{\langle\xi\rangle^2}, \quad g_{\rho,\delta} = \frac{dx^2}{\langle\xi\rangle^{-2\delta}} + \frac{d\xi^2}{\langle\xi\rangle^{2\rho}}$$

($0 \leq \delta \leq \rho \leq 1$ and $\delta < 1$) and some related symbol classes

$$a \in S(\langle\xi\rangle^m, g_{\text{class}}) = S^m \quad \Leftrightarrow \quad \partial_x^\alpha \partial_\xi^\beta a(x,\xi) = O(\langle\xi\rangle^{m-|\beta|})$$

$$a \in S(\langle x\rangle^\mu \langle\xi\rangle^m, g_{\text{scat}}) \quad \Leftrightarrow \quad \partial_x^\alpha \partial_\xi^\beta a(x,\xi) = O(\langle x\rangle^{\mu-|\alpha|}\langle\xi\rangle^{m-|\beta|})$$

$$a \in S(\langle\xi\rangle^m, g_{\rho,\delta}) =: S^m_{\rho,\delta} \quad \Leftrightarrow \quad \partial_x^\alpha \partial_\xi^\beta a(x,\xi) = O(\langle\xi\rangle^{m+\delta|\alpha|-\rho|\beta|})$$

ΨDO (part 2): More general classes

There is a general notion of Riemannian metrics for pseudo-differential calculus (Hörmander). We just give a few examples

$$g_{\text{class}} = dx^2 + \frac{d\xi^2}{\langle\xi\rangle^2}, \quad g_{\text{scat}} = \frac{dx^2}{\langle x\rangle^2} + \frac{d\xi^2}{\langle\xi\rangle^2}, \quad g_{\rho,\delta} = \frac{dx^2}{\langle\xi\rangle^{-2\delta}} + \frac{d\xi^2}{\langle\xi\rangle^{2\rho}}$$

($0 \leq \delta \leq \rho \leq 1$ and $\delta < 1$) and some related symbol classes

$$a \in S(\langle\xi\rangle^m, g_{\text{class}}) = S^m \quad \Leftrightarrow \quad \partial_x^\alpha \partial_\xi^\beta a(x,\xi) = O(\langle\xi\rangle^{m-|\beta|})$$

$$a \in S(\langle x\rangle^\mu \langle\xi\rangle^m, g_{\text{scat}}) \quad \Leftrightarrow \quad \partial_x^\alpha \partial_\xi^\beta a(x,\xi) = O(\langle x\rangle^{\mu-|\alpha|}\langle\xi\rangle^{m-|\beta|})$$

$$a \in S(\langle\xi\rangle^m, g_{\rho,\delta}) =: S^m_{\rho,\delta} \quad \Leftrightarrow \quad \partial_x^\alpha \partial_\xi^\beta a(x,\xi) = O(\langle\xi\rangle^{m+\delta|\alpha|-\rho|\beta|})$$

ΨDO (part 2): More general classes

There is a general notion of Riemannian metrics for pseudo-differential calculus (Hörmander). We just give a few examples

$$g_{\text{class}} = dx^2 + \frac{d\xi^2}{\langle\xi\rangle^2}, \quad g_{\text{scat}} = \frac{dx^2}{\langle x\rangle^2} + \frac{d\xi^2}{\langle\xi\rangle^2}, \quad g_{\rho,\delta} = \frac{dx^2}{\langle\xi\rangle^{-2\delta}} + \frac{d\xi^2}{\langle\xi\rangle^{2\rho}}$$

($0 \leq \delta \leq \rho \leq 1$ and $\delta < 1$) and some related symbol classes

$$a \in S(\langle\xi\rangle^m, g_{\text{class}}) = S^m \quad \Leftrightarrow \quad \partial_x^\alpha \partial_\xi^\beta a(x,\xi) = O(\langle\xi\rangle^{m-|\beta|})$$

$$a \in S(\langle x\rangle^\mu \langle\xi\rangle^m, g_{\text{scat}}) \quad \Leftrightarrow \quad \partial_x^\alpha \partial_\xi^\beta a(x,\xi) = O(\langle x\rangle^{\mu-|\alpha|} \langle\xi\rangle^{m-|\beta|})$$

$$a \in S(\langle\xi\rangle^m, g_{\rho,\delta}) =: S^m_{\rho,\delta} \quad \Leftrightarrow \quad \partial_x^\alpha \partial_\xi^\beta a(x,\xi) = O(\langle\xi\rangle^{m+\delta|\alpha|-\rho|\beta|})$$

ΨDO (part 2): More general classes

There is a general notion of Riemannian metrics for pseudo-differential calculus (Hörmander). We just give a few examples

$$g_{\text{class}} = dx^2 + \frac{d\xi^2}{\langle\xi\rangle^2}, \quad g_{\text{scat}} = \frac{dx^2}{\langle x\rangle^2} + \frac{d\xi^2}{\langle\xi\rangle^2}, \quad g_{\rho,\delta} = \frac{dx^2}{\langle\xi\rangle^{-2\delta}} + \frac{d\xi^2}{\langle\xi\rangle^{2\rho}}$$

($0 \leq \delta \leq \rho \leq 1$ and $\delta < 1$) and some related symbol classes

$$a \in S(\langle\xi\rangle^m, g_{\text{class}}) = S^m \quad \Leftrightarrow \quad \partial_x^\alpha \partial_\xi^\beta a(x,\xi) = O(\langle\xi\rangle^{m-|\beta|})$$

$$a \in S(\langle x\rangle^\mu \langle\xi\rangle^m, g_{\text{scat}}) \quad \Leftrightarrow \quad \partial_x^\alpha \partial_\xi^\beta a(x,\xi) = O(\langle x\rangle^{\mu-|\alpha|} \langle\xi\rangle^{m-|\beta|})$$

$$a \in S(\langle\xi\rangle^m, g_{\rho,\delta}) =: S^m_{\rho,\delta} \quad \Leftrightarrow \quad \partial_x^\alpha \partial_\xi^\beta a(x,\xi) = O(\langle\xi\rangle^{m+\delta|\alpha|-\rho|\beta|})$$

ΨDO (part 2): More general classes

There is a general notion of Riemannian metrics for pseudo-differential calculus (Hörmander). We just give a few examples

$$g_{\text{class}} = dx^2 + \frac{d\xi^2}{\langle\xi\rangle^2}, \quad g_{\text{scat}} = \frac{dx^2}{\langle x\rangle^2} + \frac{d\xi^2}{\langle\xi\rangle^2}, \quad g_{\rho,\delta} = \frac{dx^2}{\langle\xi\rangle^{-2\delta}} + \frac{d\xi^2}{\langle\xi\rangle^{2\rho}}$$

($0 \le \delta \le \rho \le 1$ and $\delta < 1$) and some related symbol classes

$$a \in S(\langle\xi\rangle^m, g_{\text{class}}) = S^m \quad \Leftrightarrow \quad \partial_x^\alpha \partial_\xi^\beta a(x,\xi) = O(\langle\xi\rangle^{m-|\beta|})$$

$$a \in S(\langle x\rangle^\mu \langle\xi\rangle^m, g_{\text{scat}}) \quad \Leftrightarrow \quad \partial_x^\alpha \partial_\xi^\beta a(x,\xi) = O(\langle x\rangle^{\mu-|\alpha|} \langle\xi\rangle^{m-|\beta|})$$

$$a \in S(\langle\xi\rangle^m, g_{\rho,\delta}) =: S^m_{\rho,\delta} \quad \Leftrightarrow \quad \partial_x^\alpha \partial_\xi^\beta a(x,\xi) = O(\langle\xi\rangle^{m+\delta|\alpha|-\rho|\beta|})$$

ΨDO (part 2): More general classes

There is a general notion of Riemannian metrics for pseudo-differential calculus (Hörmander). We just give a few examples

$$g_{\text{class}} = dx^2 + \frac{d\xi^2}{\langle\xi\rangle^2}, \quad g_{\text{scat}} = \frac{dx^2}{\langle x\rangle^2} + \frac{d\xi^2}{\langle\xi\rangle^2}, \quad g_{\rho,\delta} = \frac{dx^2}{\langle\xi\rangle^{-2\delta}} + \frac{d\xi^2}{\langle\xi\rangle^{2\rho}}$$

($0 \leq \delta \leq \rho \leq 1$ and $\delta < 1$) and some related symbol classes

$$a \in S(\langle\xi\rangle^m, g_{\text{class}}) = S^m \quad \Leftrightarrow \quad \partial_x^\alpha \partial_\xi^\beta a(x,\xi) = O(\langle\xi\rangle^{m-|\beta|})$$

$$a \in S(\langle x\rangle^\mu \langle\xi\rangle^m, g_{\text{scat}}) \quad \Leftrightarrow \quad \partial_x^\alpha \partial_\xi^\beta a(x,\xi) = O(\langle x\rangle^{\mu-|\alpha|}\langle\xi\rangle^{m-|\beta|})$$

$$a \in S(\langle\xi\rangle^m, g_{\rho,\delta}) =: S^m_{\rho,\delta} \quad \Leftrightarrow \quad \partial_x^\alpha \partial_\xi^\beta a(x,\xi) = O(\langle\xi\rangle^{m+\delta|\alpha|-\rho|\beta|})$$

ΨDO (part 2): More general classes

There is a general notion of Riemannian metrics for pseudo-differential calculus (Hörmander). We just give a few examples

$$g_{\text{class}} = dx^2 + \frac{d\xi^2}{\langle\xi\rangle^2}, \quad g_{\text{scat}} = \frac{dx^2}{\langle x\rangle^2} + \frac{d\xi^2}{\langle\xi\rangle^2}, \quad g_{\rho,\delta} = \frac{dx^2}{\langle\xi\rangle^{-2\delta}} + \frac{d\xi^2}{\langle\xi\rangle^{2\rho}}$$

($0 \le \delta \le \rho \le 1$ and $\delta < 1$) and some related symbol classes

$$a \in S(\langle\xi\rangle^m, g_{\text{class}}) = S^m \quad \Leftrightarrow \quad \partial_x^\alpha \partial_\xi^\beta a(x,\xi) = O(\langle\xi\rangle^{m-|\beta|})$$

$$a \in S(\langle x\rangle^\mu \langle\xi\rangle^m, g_{\text{scat}}) \quad \Leftrightarrow \quad \partial_x^\alpha \partial_\xi^\beta a(x,\xi) = O(\langle x\rangle^{\mu-|\alpha|}\langle\xi\rangle^{m-|\beta|})$$

$$a \in S(\langle\xi\rangle^m, g_{\rho,\delta}) =: S^m_{\rho,\delta} \quad \Leftrightarrow \quad \partial_x^\alpha \partial_\xi^\beta a(x,\xi) = O(\langle\xi\rangle^{m+\delta|\alpha|-\rho|\beta|})$$

ΨDO (part 2): More general classes

There is a general notion of Riemannian metrics for pseudo-differential calculus (Hörmander). We just give a few examples

$$g_{\text{class}} = dx^2 + \frac{d\xi^2}{\langle\xi\rangle^2}, \quad g_{\text{scat}} = \frac{dx^2}{\langle x\rangle^2} + \frac{d\xi^2}{\langle\xi\rangle^2}, \quad g_{\rho,\delta} = \frac{dx^2}{\langle\xi\rangle^{-2\delta}} + \frac{d\xi^2}{\langle\xi\rangle^{2\rho}}$$

($0 \le \delta \le \rho \le 1$ and $\delta < 1$) and some related symbol classes

$$a \in S(\langle\xi\rangle^m, g_{\text{class}}) = S^m \quad \Leftrightarrow \quad \partial_x^\alpha \partial_\xi^\beta a(x,\xi) = O(\langle\xi\rangle^{m-|\beta|})$$

$$a \in S(\langle x\rangle^\mu \langle\xi\rangle^m, g_{\text{scat}}) \quad \Leftrightarrow \quad \partial_x^\alpha \partial_\xi^\beta a(x,\xi) = O(\langle x\rangle^{\mu-|\alpha|}\langle\xi\rangle^{m-|\beta|})$$

$$a \in S(\langle\xi\rangle^m, g_{\rho,\delta}) =: S^m_{\rho,\delta} \quad \Leftrightarrow \quad \partial_x^\alpha \partial_\xi^\beta a(x,\xi) = O(\langle\xi\rangle^{m+\delta|\alpha|-\rho|\beta|})$$

ΨDO (part 2): Other quantizations

The **Weyl quantization** of a symbol a is defined by

$$Op_h^W(a)u(x) = (2\pi)^{-n} \int\int e^{i(x-y)\cdot\xi} a\left(\frac{x+y}{2}, h\xi\right) u(y)dyd\xi.$$

More generally, one sometimes considers for $t \in [0,1]$

$$Op_{h,t}(a)u(x) = (2\pi)^{-n} \int\int e^{i(x-y)\cdot\xi} a\left(tx + (1-t)y, h\xi\right) u(y)dyd\xi.$$

All these quantizations can be written using Op_h, i.e.

$$Op_{h,t}(a) = Op_h(a_t(h)),$$

for some $(a_{t,h})_{h\in(0,1]}$ in the same class as a.

ΨDO (part 2): Other quantizations

The **Weyl quantization** of a symbol *a* is defined by

$$Op_h^W(a)u(x) = (2\pi)^{-n} \int\int e^{i(x-y)\cdot\xi} a\left(\frac{x+y}{2}, h\xi\right) u(y)dyd\xi.$$

More generally, one sometimes considers for $t \in [0, 1]$

$$Op_{h,t}(a)u(x) = (2\pi)^{-n} \int\int e^{i(x-y)\cdot\xi} a\left(tx + (1-t)y, h\xi\right) u(y)dyd\xi.$$

All these quantizations can be written using Op_h, i.e.

$$Op_{h,t}(a) = Op_h(a_t(h)),$$

for some $(a_{t,h})_{h\in(0,1]}$ in the same class as *a*.

ΨDO (part 2): Other quantizations

The **Weyl quantization** of a symbol *a* is defined by

$$Op_h^W(a)u(x) = (2\pi)^{-n} \int\int e^{i(x-y)\cdot\xi} a\left(\frac{x+y}{2}, h\xi\right) u(y)dyd\xi.$$

More generally, one sometimes considers for $t \in [0,1]$

$$Op_{h,t}(a)u(x) = (2\pi)^{-n} \int\int e^{i(x-y)\cdot\xi} a\left(tx + (1-t)y, h\xi\right) u(y)dyd\xi.$$

All these quantizations can be written using Op_h, i.e.

$$Op_{h,t}(a) = Op_h(a_t(h)),$$

for some $(a_{t,h})_{h\in(0,1]}$ in the same class as *a*.

ΨDO (part 2): Symbolic calculus and L^2 bounds revisited

Theorem. Let m_1, m_2 be order functions and $a \in S(m_1, g_0)$, $b \in S(m_2, g_0)$.

- **Composition.** $Op_h^W(a)Op_h^W(b) = Op_h^W(c(h))$ with $c(h) \in S(m_1 m_2, g_0)$ of the form

$$c(h)(x,\xi) \sim \sum_j \left(\frac{h}{2i}\right)^j (\partial_y \cdot \partial_\xi - \partial_x \cdot \partial_\eta)^j a(x,\xi) b(y,\eta)\Big|_{y=x,\ \eta=\xi}$$

- **(Formal) Adjoint.**

$$Op_h^W(a)^* = Op_h^W(\overline{a}).$$

Theorem (Calderón-Vaillancourt). There are $C > 0$ and $N \geq 0$ such that for all $a \in S(1, g_0)$ and all $h \in (0,1]$

$$||Op_h^W(a)||_{L^2 \to L^2} \leq C \max_{|\alpha+\beta| \leq N} ||\partial_x^\alpha \partial_\xi^\beta a||_{L^\infty}.$$

ΨDO (part 2): The Gårding inequality

Theorem (Gårding inequality).

- **(Semiclassical version).** If $a \in S(1, g_0)$ satisfies $a \geq 0$, then there exists $C > 0$ such that

$$\mathrm{Re}(Op_h(a)u, u) \geq -Ch||u||^2, \qquad h \in (0, 1],\ u \in \mathcal{S}.$$

- **("Classical" version)** If $a \in S^1 = S(\langle \xi \rangle, g_{\mathrm{class}})$ satisfies $a \geq 0$, then there exists $C > 0$ such that

$$\mathrm{Re}(Op_1(a)u, u) \geq -C||u||^2, \qquad u \in \mathcal{S}.$$

Remark. Works for systems of ΨDO too. In the purely scalar case, there is the Fefferman-Phong inequality ($h \to h^2$).

ΨDO (part 2): The Gårding inequality

Theorem (Gårding inequality).

- **(Semiclassical version).** If $a \in S(1, g_0)$ satisfies $a \geq 0$, then there exists $C > 0$ such that

$$\mathrm{Re}(Op_h(a)u, u) \geq -Ch||u||^2, \qquad h \in (0, 1],\ u \in \mathcal{S}.$$

- **(“Classical” version)** If $a \in S^1 = S(\langle \xi \rangle, g_{\text{class}})$ satisfies $a \geq 0$, then there exists $C > 0$ such that

$$\mathrm{Re}(Op_1(a)u, u) \geq -C||u||^2, \qquad u \in \mathcal{S}.$$

Remark. Works for systems of ΨDO too. In the purely scalar case, there is the Fefferman-Phong inequality ($h \to h^2$).

ΨDO (part 2): The Gårding inequality

Theorem (Gårding inequality).

- **(Semiclassical version).** If $a \in S(1, g_0)$ satisfies $a \geq 0$, then there exists $C > 0$ such that

$$\mathrm{Re}(Op_h(a)u, u) \geq -Ch||u||^2, \qquad h \in (0, 1],\ u \in \mathcal{S}.$$

- **("Classical" version)** If $a \in S^1 = S(\langle \xi \rangle, g_{\mathrm{class}})$ satisfies $a \geq 0$, then there exists $C > 0$ such that

$$\mathrm{Re}(Op_1(a)u, u) \geq -C||u||^2, \qquad u \in \mathcal{S}.$$

Remark. Works for systems of ΨDO too. In the purely scalar case, there is the Fefferman-Phong inequality ($h \to h^2$).

ΨDO (part 2): The Gårding inequality

Theorem (Gårding inequality).

- **(Semiclassical version).** If $a \in S(1, g_0)$ satisfies $a \geq 0$, then there exists $C > 0$ such that

$$\mathrm{Re}(Op_h(a)u, u) \geq -Ch||u||^2, \qquad h \in (0,1],\ u \in \mathcal{S}.$$

- **("Classical" version)** If $a \in S^1 = S(\langle\xi\rangle, g_{\text{class}})$ satisfies $a \geq 0$, then there exists $C > 0$ such that

$$\mathrm{Re}(Op_1(a)u, u) \geq -C||u||^2, \qquad u \in \mathcal{S}.$$

Remark. Works for systems of ΨDO too. In the purely scalar case, there is the Fefferman-Phong inequality ($h \to h^2$).

ΨDO (part 2): The Gårding inequality

Theorem (Gårding inequality).

- **(Semiclassical version).** If $a \in S(1, g_0)$ satisfies $a \geq 0$, then there exists $C > 0$ such that

$$\mathrm{Re}(Op_h(a)u, u) \geq -Ch||u||^2, \qquad h \in (0, 1],\ u \in \mathcal{S}.$$

- **("Classical" version)** If $a \in S^1 = S(\langle \xi \rangle, g_{\mathrm{class}})$ satisfies $a \geq 0$, then there exists $C > 0$ such that

$$\mathrm{Re}(Op_1(a)u, u) \geq -C||u||^2, \qquad u \in \mathcal{S}.$$

Remark. Works for systems of ΨDO too. In the purely scalar case, there is the Fefferman-Phong inequality ($h \to h^2$).

ΨDO (part 2): Proof of the Gårding inequality (semiclassical version)

For $(q,p) \in \mathbb{R}^{2n}$, define the (unitary) **phase space translation operator**

$$U(q,p)f(x) = e^{ix\cdot p}f(x-q).$$

Pick then $\eta \in \mathcal{S}$ such that $||\eta||_{L^2} = 1$ and define the **wave packets** $(\eta_{q,p})_{(q,p)\in\mathbb{R}^{2n}}$

$$\eta_{q,p}(x) = U(q,p)\eta(x).$$

Define the Bargmann transform

$$(Bu)(q,p) = (\eta_{q,p}, u)_{L^2}, \qquad u \in \mathcal{S}.$$

Lemma. B maps $\mathcal{S}(\mathbb{R}^n)$ into $\mathcal{S}(\mathbb{R}^{2n})$.
One also has

$$(B^*\Psi)(x) = \iint \Psi(q,p)\eta_{q,p}(x)dqdp.$$

Proposition (resolution of identity). $B^*B = (2\pi)^n I$.

ΨDO (part 2): Proof of the Gårding inequality (semiclassical version)

The resolution of identity $B^*B = (2\pi)^n I$ allows to define the **anti-Wick quantization**: given $a \in L^\infty(\mathbb{R}^{2n})$, we let M_a be the multiplication operator by a, and

$$Op^{aW}(a) = (2\pi)^{-n} B^* M_a B.$$

Using that

$$||(2\pi)^{-n/2} Bu||_{L^2(\mathbb{R}^{2n})} = ||u||_{L^2(\mathbb{R}^n)},$$

one has

$$||Op^{aW}(a)||_{L^2 \to L^2} \leq ||a||_{L^\infty}.$$

The main property is the obvious fact that

$$a \geq 0 \qquad \Longrightarrow \qquad Op^{aW}(a) \geq 0.$$

Important observation: all this holds for *any* profil η.

ΨDO (part 2): Proof of the Gårding inequality (semiclassical version)

The resolution of identity $B^*B = (2\pi)^n I$ allows to define the **anti-Wick quantization**: given $a \in L^\infty(\mathbb{R}^{2n})$, we let M_a be the multiplication operator by a, and

$$Op^{aW}(a) = (2\pi)^{-n} B^* M_a B.$$

Using that

$$||(2\pi)^{-n/2} Bu||_{L^2(\mathbb{R}^{2n})} = ||u||_{L^2(\mathbb{R}^n)},$$

one has

$$||Op^{aW}(a)||_{L^2 \to L^2} \leq ||a||_{L^\infty}.$$

The main property is the obvious fact that

$$a \geq 0 \qquad \Longrightarrow \qquad Op^{aW}(a) \geq 0.$$

Important observation: all this holds for *any* profil η.

ΨDO (part 2): Proof of the Gårding inequality (semiclassical version)

Assume that η is *even* and *real valued* (e.g. Gaussian) and consider

$$\eta_h(x) = h^{-\frac{n}{4}}\eta(x/h^{\frac{1}{2}}), \qquad \mathcal{W}_h(x,\xi) = (2\pi h)^{-n} e^{ix\cdot\xi}\eta(x/h^{\frac{1}{2}})\hat{\eta}(\xi/h^{\frac{1}{2}})$$

($\mathcal{W}_h$ is normalized in $L^1(\mathbb{R}^{2n})$). Then

Proposition. $Op^{aW}(a) = Op_h(a * \mathcal{W}_h)$

Proof of the theorem.

$$\begin{aligned} Op_h(a) &= Op^{aW}(a) - Op^{aW}(a) + Op_h(a) \\ &= \underbrace{Op^{aW}(a)}_{\geq 0} + \underbrace{Op_h(a - a * \mathcal{W}_h)}_{=O_{L^2\to L^2}(h)} \end{aligned}$$

using (for the $O(h)$) the Calderón-Vaillancourt theorem and the fact that

$$||\partial_x^\alpha \partial_\xi^\beta (a - a * \mathcal{W}_h)||_{L^\infty} \lesssim h.$$

ΨDO (part 2): The Beals Theorem

We have already mentioned the (exact) formulas

$$[x_j, Op_h(a)] = ihOp_h(\partial_{\xi_j} a), \qquad [h\partial_j, Op_h(a)] = hOp_h(\partial_{x_j} a).$$

They still hold for the Weyl calculus. Setting $ad_B A := [B, A]$ then, if $A_h = Op_h^W(a)$ for some $a \in S(1, g_0)$, for any N and

$$B_k \in \{Op_h^W(x_1), \cdots, Op_h^W(x_n), Op_h^W(\xi_1), \ldots, Op_h^W(\xi_n)\}$$

one has

$$ad_{B_1} \cdots ad_{B_N} A_h = h^N Op_h^W(\underbrace{a_N}_{\in S(1,g_0)}) = O_{L^2 \to L^2}(h^N).$$

Theorem. Let $A_h : \mathcal{S} \to \mathcal{S}'$ be a family of continuous operators. Then $A_h = Op_h^W(a(h))$ for some bounded family $(a(h))_{h\in(0,1]}$ of $S(1, g_0)$ iff for all N and all B_k as above

$$ad_{B_1} \cdots ad_{B_N} A_h = O_{L^2 \to L^2}(h^N).$$

An example of application of the Beals theorem

Proposition. *Assume that m is an order function and that* $m \in S(m, g_0)$*. Then, if h is small enough,* $Op_h(a)$ *has an inverse which is a* ΨDO*, i.e. there exists a bounded family* $(\tilde{m}_h)_{h \ll 1}$ *in* $S(1/m, g_0)$ *such that*

$$Op_h^W(\tilde{m}_h) Op_h^W(m) = I = Op_h(1).$$

Proof. Observe that $1/m$ is an order function. Compute then

$$Op_h^W(1/m) Op_h^W(m) = I + h Op_h^W(b(h)) =: A_h,$$

with $(b(h))_{h \in (0,1]}$ bdd in $S(1, g_0)$. By Calderón-Vaillancourt, A_h is invertible for $h \ll 1$, and $||A_h^{-1}|| \to 1$ as $h \to 0$. By iteration of

$$[B_k, A_h^{-1}] = -A_h^{-1} [B_k, A_h] A_h^{-1} = O_{L^2 \to L^2}(h)$$

the Beals Theorem gives $A_h^{-1} = Op_h^W(a(h))$ for some $(a(h))_{h \in (0,1]}$ bounded in $S(1, g_0)$. We then get result with $Op_h^W(\tilde{m}_h) := Op_h^W(a(h)) Op_h^W(1/m)$. □

Quantum mechanical interpretation and some dynamics

Classical mechanics: particles living in $\mathbb{R}^n_x$ with motion described in the **phase space** $\mathbb{R}^{2n} = \mathbb{R}^n_x \times \mathbb{R}^n_\xi$, e.g.

$$m\ddot{x}(t) = -\nabla V(x(t))$$

reads equivalently

$$\begin{cases} \dot{x}(t) & = \frac{\xi(t)}{m}, \\ \dot{\xi}(t) & = (\nabla V)(x(t)) \end{cases} \qquad \text{or} \qquad \frac{d}{dt}(x(t), \xi(t)) = X_H(x(t), \xi(t))$$

where X_H is the **Hamiltonian vector field** associated with the (energy) **Hamiltonian** H

$$H(x,\xi) = \frac{1}{2m}|\xi|^2 + V(x), \qquad X_H = \left(\nabla_\xi H, -\nabla_x H\right) = \left(\frac{\xi}{m}, -\nabla V\right)$$

Quantum mechanical interpretation and some dynamics

We denote the (Hamiltonian) flow of X_H by Φ_H^t, i.e.

$$\Phi_H^t(y,\eta) = (x(t),\xi(t)) \qquad \text{with} \qquad \begin{cases} \frac{d}{dt}(x(t),\xi(t)) & = X_H(x(t),\xi(t)) \\ (x(0),\xi(0)) & = (y,\eta) \end{cases}$$

(we assume it is complete, e.g. if $\nabla^2 V$ bounded). Other interesting types of H include

$$H(x,\xi) = \sum_{j,k=1}^{n} g^{jk}(x)\xi_j\xi_k$$

with $(g^{jk}(x))$ the coefficients of the co-metric tensor associated to a Riemannian metric $\rightarrow$ **geodesic flow.**

Quantum mechanical interpretation and some dynamics

Functions $a : \mathbb{R}^{2n} \to \mathbb{R}$ are called **classical observables.**
For instance

- $a(x, \xi) = x_j$ (position in the j-th direction)
- $a(x, \xi) = \xi_k$ (k-th component of the momentum)
- $a(x, \xi) = H(x, \xi)$ (total energy)
- many others...

One may want to "observe" such quantities along the dynamics, i.e. consider $a \circ \Phi_H^t$.

Quantum mechanical interpretation and some dynamics

Quantum mechanics: particles living in $\mathbb{R}^n_x$ are now described by **wave functions** $\psi \in L^2(\mathbb{R}^n, \mathbb{C})$ and the motion by the **Schrödinger equation**

$$\begin{cases} i\hbar\partial_t\psi(t) & = \hat{H}_\hbar\psi(t) \\ \psi(0) & = \psi_0 \end{cases} \qquad (\hbar \sim \text{ Planck's constant})$$

where $\hat{H}_\hbar$, the **quantum Hamiltonian**, is a selfadjoint operator on L^2 e.g.

$$\hat{H}_\hbar = -\frac{\hbar^2}{2m}\Delta_x + V(x).$$

This operator is obtained by **quantizing** the classical Hamiltonian $H(x,\xi) = \frac{|\xi|^2}{2m} + V(x)$, i.e.

$$\hat{H}_\hbar = Op_\hbar(H).$$

Quantum mechanical interpretation and some dynamics

Quantum observables are now operators on L^2, which can be obtained by quantizing classical observables=symbols, i.e.

$$\hat{A}_\hbar = Op_\hbar(a).$$

ΨDO provide a way to quantize classical observables. Note that the **composition theorem** of ΨDO gives

$$\begin{aligned}
\left[Op_\hbar(a), Op_\hbar(b)\right] &= Op_\hbar(a)Op_\hbar(b) - Op_\hbar(b)Op_\hbar(a) \\
&= Op_\hbar\Big(ab + \frac{\hbar}{i}\nabla_\xi a \cdot \nabla_x b + O(\hbar^2)\Big) - \\
&\quad Op_\hbar\Big(ba + \frac{\hbar}{i}\nabla_\xi b \cdot \nabla_x a + O(\hbar^2)\Big) \\
&= \frac{\hbar}{i} Op_\hbar(\{a, b\}) + O(\hbar^2)
\end{aligned}$$

where $\{a, b\}$ is the **Poisson bracket** of a and b. This is an aspect of the **uncertainty principle** of quantum mechanics.

Quantum mechanical interpretation and some dynamics

The quantum flow is described by the **unitary** group $U_\hbar(t) = e^{-i\frac{t}{\hbar}\hat{H}_\hbar}$ and we can "observe" quantities like

$$\left(U_\hbar(t)\psi_0, \hat{A}_\hbar U_\hbar(t)\psi_0\right) = \left(\psi_0, \underbrace{U_\hbar(-t)\hat{A}_\hbar U_\hbar(t)}_{\hat{A}_\hbar(t)}\psi_0\right).$$

Theorem. For $a \in C_0^\infty(\mathbb{R}^{2n})$ and reasonable V, if $\hat{A}_\hbar = Op_\hbar(a)$ then for all (fixed) t,

$$\hat{A}_\hbar(t) = Op_\hbar(a \circ \Phi_H^t) + O_{L^2 \to L^2}(\hbar)$$

Interpretation: In the semiclassical limit $\hbar \to 0$, the Schrödinger equation converges to the **Liouville equation**

$$\partial_t a(t) - \{H, a(t)\} = \partial_t a(t) - X_H \cdot \nabla_{x,\xi} a(t) = 0.$$

Correspondence between the classical Φ_H^t and quantum $U_\hbar(t)$ flows (vs Brownian motion for heat like semigroups)

Quantum mechanical interpretation and some dynamics

The quantum flow is described by the **unitary** group $U_\hbar(t) = e^{-i\frac{t}{\hbar}\hat{H}_\hbar}$ and we can "observe" quantities like

$$\left(U_\hbar(t)\psi_0, \hat{A}_\hbar U_\hbar(t)\psi_0\right) = \left(\psi_0, \underbrace{U_\hbar(-t)\hat{A}_\hbar U_\hbar(t)}_{\hat{A}_\hbar(t)}\psi_0\right).$$

Theorem. For $a \in C_0^\infty(\mathbb{R}^{2n})$ and reasonable V, if $\hat{A}_\hbar = Op_\hbar(a)$ then for all (fixed) t,

$$\hat{A}_\hbar(t) = Op_\hbar(a \circ \Phi_H^t) + O_{L^2\to L^2}(\hbar)$$

Interpretation: In the semiclassical limit $\hbar \to 0$, the Schrödinger equation converges to the **Liouville equation**

$$\partial_t a(t) - \{H, a(t)\} = \partial_t a(t) - X_H \cdot \nabla_{x,\xi} a(t) = 0.$$

Correspondence between the classical Φ_H^t and quantum $U_\hbar(t)$ flows (vs Brownian motion for heat like semigroups)

Semiclassical measures

Assume that $(\hbar_k)$ is a (positive) sequence going to 0 and that we are given a sequence (ψ_k) of normalized functions in L^2. Define the sequence of distributions $(\mu_k) \in \mathcal{D}'(\mathbb{R}^{2n})$ by

$$\langle \mu_k, a \rangle := (\psi_k, Op_{\hbar_k}(a)\psi_k).$$

These are distributions since $|\langle \mu_k, a \rangle| \leq ||Op_{\hbar_k}(a)||_{L^2 \to L^2}$ is controlled by some seminorm of a in $S(1, g_0)$.
Assume then that, up to possible extraction, μ_k converges to some μ.

Proposition. The limit μ is a probability measure on $\mathbb{R}^{2n}$ (called **semiclassical measure** associated to (ψ_k)).

Proof. To get a measure, it suffices to show $a \geq 0 \Rightarrow \langle \mu, a \rangle \geq 0$. This follows from the Gårding inequality since, if $a \geq 0$,

$$\langle \mu_k, a \rangle \geq -C_a h_k ||\psi_k||^2 \to 0, \qquad k \to \infty.$$

It is a probability measure since $\langle \mu_k, 1 \rangle = ||\psi_k||^2 = 1.$ $\square$

Invariant semiclassical measures

Assume that (ψ_k) is a sequence of (normalized) *eigenfunctions* of $\hat{H}_{\hbar_k} = Op_{\hbar_k}(H)$ (say with $H(x,\xi) = |\xi|^2/2m + V(x)$).

Proposition. Any semiclassical measure μ associated to (ψ_k) is invariant by the classical Hamiltonian flow:

$$\langle \mu, a \circ \Phi_H^t \rangle = \langle \mu, a \rangle, \text{for all } a \text{ and } t.$$

Proof. Up to extraction

$$\begin{aligned}
\langle \mu, a \rangle &= \lim_{k\to\infty} (\psi_k, Op_{\hbar_k}(a)\psi_k) \\
&= \lim_{k\to\infty} (U_{\hbar_k}(t)\psi_k, Op_{\hbar_k}(a)U_{\hbar_k}(t)\psi_k) \\
&= \lim_{k\to\infty} (\psi_k, U_{\hbar_k}(-t)Op_{\hbar_k}(a)U_{\hbar_k}(t)\psi_k) \\
&= \lim_{k\to\infty} (\psi_k, Op_{\hbar_k}(a \circ \Phi_H^t)\psi_k) + O(\hbar_k) = \langle \mu, a \circ \Phi_H^t \rangle \quad \square
\end{aligned}$$

Interest: describe how/where eigenfunctions are “localized” in the classical phase space.