Kinetic Models — Lecture 1
From Newton's Equations
to Collisionless Kinetic Models

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What is a Kinetic Model?

System of N identical particles, with pairwise interactions; $N \gg 1$ (e.g. $N =$ Avogadro number $\simeq 6.02 \cdot 10^{23}$...)

Dynamics described

(a) either by the system of Newton's motion equations for each particle
(b) or by the motion equation for the "typical particle" driven by the collective interaction with all the other particles

Approach (b) is usually referred to as a kinetic model

Advantages/Drawbacks

(a) Perfect in theory, unfeasible in practice (phase space of high dimension $6N$, how to measure/observe initial data/trajectories?)
(b) Only an approximation, but set on a phase space of low (fixed) dimension 6

Problems
(1) To justify approach (b) by a rigorous derivation from (a), possibly with a convergence rate as the particle number $N \to \infty$
(2) To study the mathematical structure of kinetic models (conservation laws, equilibrium solutions, regularity...)

Examples of kinetic models are
•the Vlasov-Poisson or Vlasov-Maxwell system used in the modeling of plasmas or ionized gases
•the Boltzmann or the Landau equations used in the kinetic theory of gases or plasmas

Outline

Lecture 1 From Newton's Equations to Collisionless Kinetic Models

Lecture 2 Examples of Collisional Kinetic Models

Lecture 3 The Regularity Problem for the Landau Equation

The N-Body Problem in Classical Mechanics

System of N identical point particles of mass m, spatial domain $\mathbf{R}^d$

Newton's second law for the motion of the kth particle:

$$m\dot{x}_j = \xi_j\,, \qquad \dot{\xi}_j = \sum_{\substack{k=1\\k\neq j}}^{N} -\nabla \underbrace{V(x_j - x_k)}_{\text{interaction potential}}\,, \qquad 1 \le j \le N$$

Assumptions on V

(H1) $\quad V(z) = V(-z) \quad$ for all $z \in \mathbf{R}^d$

(H2) $\quad V \in C^1(\mathbf{R}^d)$ with $\nabla V \in L^\infty(\mathbf{R}^d) \cap \mathrm{Lip}(\mathbf{R}^d)$

Notation set $X_N := (x_1, \ldots, x_N)$ and $\Xi_N := (\xi_1, \ldots, \xi_N)$ in $\mathbf{R}^{dN}$

Solution of the differential system with initial data (X_N^{in}, Ξ_N^{in})

$$t \mapsto (X_N(t, X_N^{in}, \Xi_N^{in}), \Xi_N(t, X_N^{in}, \Xi_N^{in}))$$

Mean Field Scaling

Rescaled time, position and momentum:

$$\hat{t} = t/N\,, \quad \hat{x}_j(\hat{t}) = x_j(t)\,, \quad \hat{\xi}_j(\hat{t}) = \xi_j(t)$$

Motion equations

$$mN\frac{d\hat{x}_j}{d\hat{t}} = \hat{\xi}_j\,, \qquad N\frac{d\hat{\xi}_j}{d\hat{t}} = \sum_{\substack{k=1\\k\neq j}}^{N} -\nabla V(\hat{x}_j - \hat{x}_k)$$

Finite total mass assumption

$$Nm = 1$$

Henceforth drop hats on all variables; our starting point is

$$\dot{x}_j = \xi_j\,, \qquad \dot{\xi}_j = -\frac{1}{N}\sum_{\substack{k=1\\k\neq j}}^{N} \nabla V(x_j - x_k)$$

Vlasov Equation

Unknown $f(t, dxd\xi)$ = single-particle phase-space number density

$$(\partial_t + \xi \cdot \nabla_x) f - \nabla_x V_f \cdot \nabla_\xi f = 0\,, \qquad x, \xi \in \mathbf{R}^d$$

where $V_f \equiv V_f(t, x)$ is the **mean-field potential**

$$V_f(t, x) := \iint_{\mathbf{R}^d \times \mathbf{R}^d} V(x - y) f(t, dyd\eta) = (V \star f(t))(x, \xi)$$

Notation set of Borel probability measures on $\mathbf{R}^n$ denoted $\mathcal{P}(\mathbf{R}^n)$

$$\mu \in \mathcal{P}_k(\mathbf{R}^n) \iff \int_{\mathbf{R}^n} |x|^k \mu(dx) < \infty$$

Existence/Uniqueness For each $f^{in} \in \mathcal{P}_1(\mathbf{R}^{2d})$, there exists a unique weak solution $f \in C([0, +\infty); \mathcal{P}(\mathbf{R}^{2d}), \mathrm{dist}_{\mathrm{MK},1})$ of the Vlasov equation such that $f\big|_{t=0} = f^{in}$

The N-particle phase space **empirical measure** is

$$\mu_{(X_N,\Xi_N)(t)} := \frac{1}{N}\sum_{k=1}^{N}\delta_{x_k(t),\xi_k(t)}$$

Klimontovich Theorem The two conditions below are equivalent

(a) $t\mapsto(X_N,\Xi_N)(t)$ is a solution of Newton's differential system of motion equations

(b) $t\mapsto\mu_{(X_N,\Xi_N)(t)}$ is a weakly continuous in time, measure-valued solution of the Vlasov equation

Proof of the Klimontovich Theorem

Since $V \in C^1(\mathbf{R}^d)$ and V is even (by (H1)), ∇V is odd, so that $\nabla V(0) = 0$ and therefore

$$\frac{1}{N}\sum_{\substack{k=1\\k\neq j}}^{N} \nabla V(x_j(t) - x_k(t)) = \frac{1}{N}\sum_{k=1}^{N} \nabla V(x_j(t) - x_k(t))$$

$$= \int_{\mathbf{R}^{2d}} \nabla V(x_j(t) - z)\mu_{(X_N,\Xi_N)(t)}(dz d\zeta)$$

Thus Newton's 2nd law for the jth particle is the defining differential system of ODEs for the characteristic curves of the Vlasov equation localized at $(x_j(t), \xi_j(t))$

FROM NEWTON TO VLASOV WITH $C^{1,1}$ POTENTIALS

W. Braun, K. Hepp: Commun. Math. Phys. 56 (1977), 101–113
R.L. Dobrushin: Functional Anal. Appl. 13 (1979), 115–1223

Dobrushin's Theorem (1979)

Assume that V satisfies (H1-2). Let $f^{in} \in \mathcal{P}_1(\mathbf{R}^{2d})$, and let f be the (weak) solution of the Vlasov equation with initial data f^{in}.

Let $t \mapsto (X_N, \Xi_N)(t)$ be the solution of Newton's differential system with initial data (X_N^{in}, Ξ_N^{in}). Then

$$\mathrm{dist}_{\mathrm{MK},1}(\mu_{(X_N,\Xi_N)(t)}, f(t,\cdot)) \leq \mathrm{dist}_{\mathrm{MK},1}(\mu_{(X_N^{in},\Xi_N^{in})}, f^{in})e^{t+2\,\mathrm{Lip}(\nabla V)t}$$

Choice of initial data pick a sequence (X_N^{in}, Ξ_N^{in}) such that

$$\mathrm{dist}_{\mathrm{MK},1}(\mu_{(X_N^{in},\Xi_N^{in})}, f^{in}) \to 0 \quad \text{as } N \to \infty$$

Remark Mean-field limit $\iff$ continuous dependence of the Vlasov solution on the initial data in $\mathcal{P}_1(\mathbf{R}^{2d})$ for the metric $\mathrm{dist}_{\mathrm{MK},1}$

Assume that $f^{in}\in\mathcal{P}_q(\mathbf{R}^{2d})$ with $1<q\neq\frac{2d}{2d-1}$. Then

$$\int_{\mathbf{R}^{2dN}}\text{dist}_{\text{MK},1}(\mu_{(X_N^{in},\Xi_N^{in})},f^{in})\prod_{j=1}^N f^{in}(dx_jd\xi_j)\le CM_q^{\frac{1}{q}}\left(\frac{1}{N^{\frac{1}{q}}}+\frac{1}{N^{1-\frac{1}{q}}}\right)$$

where

$$M_q:=\iint_{\mathbf{R}^d\times\mathbf{R}^d}(|x|+|\xi|)^q f^{in}(x,\xi)dxd\xi<\infty$$

Application to the MF limit let f be the (weak) solution of the Vlasov equation with initial data f^{in}, and let $t\mapsto(X_N,\Xi_N)(t)$ be the solution of Newton's ODE system with initial data (X_N^{in},Ξ_N^{in}). Then

$$\int_{\mathbf{R}^{2dN}}\text{dist}_{\text{MK},1}(\mu_{(X_N,\Xi_N)(t,X_N^{in},\Xi_N^{in})},f(t,\cdot))\prod_{j=1}^N f^{in}(dx_jd\xi_j)$$
$$\le CM_q^{1/q}e^{t+2\,\text{Lip}(\nabla V)t}\left(N^{-\frac{1}{q}}+N^{-(1-\frac{1}{q})}\right)$$

Couplings of Probability Measures

Given $\mu, \nu \in \mathcal{P}(\mathbf{R}^n)$, a coupling of μ and ν is a probability measure $\pi \in \mathcal{P}(\mathbf{R}^n \times \mathbf{R}^n)$ such that

$$\iint_{\mathbf{R}^n \times \mathbf{R}^n} (\phi(x) + \psi(y))\pi(dxdy) = \int_{\mathbf{R}^n} \phi(x)\mu(dx) + \int_{\mathbf{R}^n} \psi(y)\nu(dy)$$

Set of couplings of μ, ν denoted $\Pi(\mu, \nu)$; obviously

$$\mu, \nu \in \mathcal{P}_p(\mathbf{R}^n) \implies \Pi(\mu, \nu) \subset \mathcal{P}_p(\mathbf{R}^n \times \mathbf{R}^n)$$

Let $p \in [1, \infty)$; for each $\mu, \nu \in \mathcal{P}_p(\mathbf{R}^n)$, the Monge-Kantorovich distance of exponent p between μ and ν is

$$\operatorname{dist}_{\mathrm{MK,p}}(\mu, \nu) = \inf_{\pi \in \Pi(\mu,\nu)} \left(\iint_{\mathbf{R}^n \times \mathbf{R}^n} |x - y|^p \pi(dxdy) \right)^{1/p}$$

Monge-Kantorovich duality

$$\operatorname{dist}_{\mathrm{MK,p}}(\mu, \nu)^p = \sup_{\substack{\phi(x)+\psi(y) \leq |x-y|^p \\ \phi,\psi \in C_b(\mathbf{R}^n)}} \left(\int_{\mathbf{R}^n} \phi(x)\mu(dx) + \int_{\mathbf{R}^n} \psi(x)\nu(dx) \right)$$

In particular

$$\operatorname{dist}_{\mathrm{MK,1}}(\mu, \nu) = \sup_{\mathrm{Lip}(\phi) \leq 1} \left| \int_{\mathbf{R}^n} \phi(z)\mu(dz) - \int_{\mathbf{R}^n} \phi(z)\nu(dz) \right|$$

Proof of Dobrushin's Inequality

Let f^{in} and $g^{in} \in \mathcal{P}_1(\mathbf{R}^{2d})$, and let f and g be the solutions of the Vlasov equation

$$\partial_t f + \{\tfrac{1}{2}|\xi|^2 + V_f(t,x), f\} = 0 \qquad f\big|_{t=0} = f^{in}$$
$$\partial_t g + \{\tfrac{1}{2}|\eta|^2 + V_g(t,y), g\} = 0 \qquad g\big|_{t=0} = g^{in}$$

Poisson bracket

$$\{\phi,\psi\}(x,\xi) = \nabla_\xi \phi(x,\xi)\cdot\nabla_x\psi(x,\xi) - \nabla_x\phi(x,\xi)\cdot\nabla_\xi\psi(x,\xi)$$

Let h be the weak solution of the Liouville equation in $\mathbf{R}^{2d}_{x,\xi}\times\mathbf{R}^{2d}_{y,\eta}$

$$\partial_t h + \{\tfrac{1}{2}|\xi|^2 + \tfrac{1}{2}|\eta|^2 + V_f(t,x) + V_g(t,y), h\} = 0\,, \quad h\big|_{t=0} = h^{in}$$

where $h^{in} \in \Pi(f^{in}, g^{in})$

Lemma 1 The weak solution $f \in C([0,+\infty), \mathcal{P}_1(\mathbf{R}^{2d}))$ satisfies

$$M_1(t) := \int_{\mathbf{R}^{2d}} (|x|+|\xi|) f(t, dx d\xi) \leq M_1(0) e^{t(\max(1,\mathrm{Lip}(\nabla V))+\mathrm{Lip}(\nabla V))}$$

Proof Multiplying both sides of the Vlasov equation by $|x|+|\xi|$, and integrating by parts

$$\dot{M}_1(t) = \int_{\mathbf{R}^{2d}} \{\tfrac{1}{2}|\xi|^2+V_f(t,x), |x|+|\xi|\} f(t, dx d\xi)$$
$$= \int_{\mathbf{R}^{2d}} (\xi \cdot \tfrac{x}{|x|} - \nabla V_f(t,x) \cdot \tfrac{\xi}{|\xi|}) f(t, dx d\xi)$$
$$\leq \int_{\mathbf{R}^{2d}} (|\xi| + |\nabla V_f(t,x)|) f(t, dx d\xi)$$

Observe that

$$
\begin{aligned}
&|\nabla_x V_f(t,x) - \nabla_x V_f(t,0)| \\
&\le \int_{\mathbf{R}^{2d}} |\nabla V(x-z) - \nabla V(-z))| f(t,dzd\zeta) \le \mathrm{Lip}(\nabla V)|x|
\end{aligned}
$$

$$
\begin{aligned}
\nabla V(0) = 0 \implies |\nabla_x V_f(t,0)| &\le \int_{\mathbf{R}^{2d}} |\nabla V(-z)| f(t,dzd\zeta) \\
&\le \mathrm{Lip}(\nabla V) \int_{\mathbf{R}^{2d}} |z| f(t,dzd\zeta) \le \mathrm{Lip}(\nabla V) M_1(t)
\end{aligned}
$$

Hence

$$
\begin{aligned}
\dot{M}_1(t) &\le \int_{\mathbf{R}^{2d}} (|\xi| + \mathrm{Lip}(\nabla V)(|x| + M_1(t))) f(t,dxd\xi) \\
&\le (\max(1, \mathrm{Lip}(\nabla V)) + \mathrm{Lip}(\nabla V)) M_1(t)
\end{aligned}
$$

Lemma 2 One has

$$h^{in}\in\Pi(f^{in},g^{in})\implies h(t)\in\Pi(f(t),g(t))\quad\text{for each } t\geq 0$$

Proof For each $\phi\in C_c^1(\mathbf{R}^{2d})$, one has

$$\frac{d}{dt}\int_{\mathbf{R}^{4d}}\phi(x,\xi)h(t,dxd\xi dyd\eta)$$

$$=\int_{\mathbf{R}^{4d}}\{\tfrac12|\xi|^2+\tfrac12|\eta|^2+V_f(t,x)+V_g(t,y),\phi(x,\xi)\}h(t,dxd\xi dyd\eta)$$

$$=\int_{\mathbf{R}^{4d}}\{\tfrac12|\xi|^2+V_f(t,x),\phi(x,\xi)\}h(t,dxd\xi dyd\eta)$$

By uniqueness for the Liouville equation with initial data f^{in} with Hamiltonian $\frac12|\xi|^2+V_f(t,x)$, this implies that

$$\int_{\mathbf{R}^{2d}}hdyd\eta=f$$

Let

$$D(t) := \int_{\mathbf{R}^{4d}} (|x-y| + |\xi-\eta|) h(t, dxd\xi dyd\eta)$$

Then

$$\dot{D}(t) = \int_{\mathbf{R}^{4d}} B(t,x,\xi,y,\eta) h(t, dxd\xi dyd\eta)$$

with

$$\begin{aligned} B(t,x,\xi,y,\eta) &= \{\tfrac{1}{2}|\xi|^2 + \tfrac{1}{2}|\eta|^2 + V_f(t,x) + V_g(t,y), |x-y| + |\xi-\eta|\} \\ &= (\xi-\eta)\cdot\tfrac{x-y}{|x-y|} - (\nabla_x V_f(t,x) - \nabla_y V_g(t,y))\cdot\tfrac{\xi-\eta}{|\xi-\eta|} \\ &\le |\xi-\eta| + |\nabla_x V_f(t,x) - \nabla_y V_g(t,y)| \end{aligned}$$

Now

$$
\begin{aligned}
&|\nabla_x V_f(t,x) - \nabla_y V_g(t,y)| \\
&\leq \int_{\mathbf{R}^{2d}} |\nabla V(x-z) - \nabla V(y-z)| f(t, dz d\zeta) \\
&+ \left| \int_{\mathbf{R}^{2d}} \nabla V(y-z) f(t, dz d\zeta) - \int_{\mathbf{R}^{2d}} \nabla V(y-z) g(t, dz d\zeta) \right| \\
&\leq \mathrm{Lip}(\nabla V)|x-y| + \mathrm{Lip}(\nabla V)\, \mathrm{dist}_{\mathrm{MK},1}(f(t), g(t))
\end{aligned}
$$

so that

$$
\begin{aligned}
B(t,x,\xi,y,\eta) \leq & |\xi - \eta| + \mathrm{Lip}(\nabla V)|x-y| \\
& + \mathrm{Lip}(\nabla V)\, \mathrm{dist}_{\mathrm{MK},1}(f(t), g(t))
\end{aligned}
$$

Hence

$$\dot{D}(t) \le \int_{\mathbf{R}^{4d}} (|\xi - \eta| + \mathrm{Lip}(\nabla V)|x - y|) h(t, dx d\xi dy d\eta)$$
$$+ \mathrm{Lip}(\nabla V)\, \mathrm{dist}_{\mathrm{MK},1}(f(t), g(t)))$$
$$\le \max(1, \mathrm{Lip}(\nabla V)) D(t) + \mathrm{Lip}(\nabla V)\, \mathrm{dist}_{\mathrm{MK},1}(f(t), g(t)))$$

One has $h(t) \in \Pi(f(t), g(t)) \implies \mathrm{dist}_{\mathrm{MK},1}(f(t), g(t)) \le D(t)$, so that

$$\dot{D}(t) \le (\max(1, \mathrm{Lip}(\nabla V)) + \mathrm{Lip}(\nabla V)) D(t)$$

and hence

$$\mathrm{dist}_{\mathrm{MK},1}(f(t), g(t))) \le D(t) \le D(0) e^{t(\max(1,\mathrm{Lip}(\nabla V)) + \mathrm{Lip}(\nabla V))}$$

Minimizing the r.h.s. in $h^{in} \in \Pi(f^{in}, g^{in})$ implies that

$$\mathrm{dist}_{\mathrm{MK},1}(f(t), g(t))) \le \mathrm{dist}_{\mathrm{MK},1}(f^{in}, g^{in}) e^{t(\max(1,\mathrm{Lip}(\nabla V)) + \mathrm{Lip}(\nabla V))}$$

Limitations of Dobrushin's Approach

•Seems limited to Lipschitz continuous interaction forces — however, can be modified to treat singular forces: see the work of Hauray-Jabin, Pickl-Lazarovici, Lazarovici...

•Convergence rate estimate limited by quantization error for the initial distribution function f^{in}

FROM NEWTON TO EULER-POISSON

S. Serfaty, Duke Math. J. **169** (2020), 2887–2935

Unknown $\rho(t,x) \geq 0$ (density) and $u(t,x) \in \mathbf{R}^3$ (velocity field)

$$\begin{cases} \partial_t \rho + \operatorname{div}_x(\rho u) = 0\,, & \rho\big|_{t=0} = \rho^{in} \\ \partial_t u + u \cdot \nabla_x u = -\nabla_x \frac{1}{|x|} \star_x \rho\,, & u\big|_{t=0} = u^{in} \end{cases}$$

If (ρ, u) is a classical solution of the pressureless Euler-Poisson system, the monokinetic distribution function

$$f(t,x,\xi) := \rho(t,x)\delta(\xi - u(t,x))\,, \qquad x, \xi \in \mathbf{R}^3$$

is a solution of the Vlasov-Poisson system

$$\begin{cases} \partial_t f + \xi \cdot \nabla_x f - \nabla_x V_f(t,x) \cdot \nabla_\xi f = 0 \\ -\Delta_x V_f(t,x) = 4\pi \int_{\mathbf{R}^3} f(t,x,\xi) d\xi \end{cases}$$

Let $u^{in} \in L^\infty(\mathbf{R}^3)$ be s.t. $\nabla_x u^{in} \in H^{2m}(\mathbf{R}^3)$, and $\rho^{in} \in H^{2m}(\mathbf{R}^3)$ s.t.

$$\rho^{in}(x) \geq 0 \text{ for a.e. } x \in \mathbf{R}^3, \quad \text{and} \quad \int_{\mathbf{R}^3} \rho^{in}(y)dy = 1$$

(1) There exists $T \equiv T[\|\rho^{in}\|_{H^{2m}(\mathbf{R}^3)} + \|\nabla_x u^{in}\|_{H^{2m}(\mathbf{R}^3)}] > 0$, and a unique solution (ρ, u) of the Euler-Poisson system s.t.

$$u \in L^\infty([0,T] \times \mathbf{R}^3) \quad \text{while } \rho \text{ and } \nabla_x u \in C([0,T], H^{2m}(\mathbf{R}^3))$$

(2) Besides, for all $t \in [0, T]$, one has

$$\rho(t,x) \geq 0 \text{ for a.e. } x \in \mathbf{R}^3, \quad \text{and} \quad \int_{\mathbf{R}^3} \rho(t,y)dy = 1$$

The Serfaty-Duerinckx Theorem (2020)

Pick $X_N^{in} = (x_1^{in}, \ldots, x_N^{in}) \in \mathbf{R}^{3N}$ such that

$$\iint_{x \neq y} \frac{(\mu_{X_N^{in}}(dx) - \rho^{in}(x)dx)(\mu_{X_N^{in}}(dy) - \rho^{in}(x)dy)}{|x-y|} \to 0$$

and pick $\Xi_N^{in} = (\xi_1^{in}, \ldots, \xi_N^{in}) \in \mathbf{R}^{3N}$ such that

$$\xi_j^{in} = u^{in}(x_j^{in}), \qquad j = 1, \ldots, N$$

Let $(X_N, \Xi_N)(t; X_N^{in}, \Xi_N^{in})$ be the solution of Newton's system of equation with Coulomb interaction. In the limit as $N \to \infty$

$$\mu_{X_N(t;X_N^{in},\Xi_N^{in})} \to \rho(t,\cdot) \text{ narrowly, and}$$

$$\frac{1}{N}\sum_{j=1}^{N} |\xi_j(t; X_N^{in}, \Xi_N^{in}) - u(t, x_j(t; X_N^{in}, \Xi_N^{in}))|^2 \to 0$$

Recalling the notation for the empirical measure

$$\mu_{Z_N} := \tfrac{1}{N}\sum_{j=1}^{N}\delta_{z_j}\,, \qquad Z_N := (z_1,\ldots,z_N)$$

for each N-tuples of positions X_N and momenta Ξ_N, each density ρ and each velocity field u, consider the modulated energy

$$\begin{aligned}\mathcal{E}[X_N,\Xi_N,\rho,u] &:= \tfrac{1}{N}\sum_{j=1}^{N}|\xi_j-u(x_j)|^2+\iint_{x\neq y}\tfrac{(\mu_{X_N}-\rho)(dx)(\mu_{X_N}-\rho)(dy)}{|x-y|}\\ &= \iint|\xi-u(x)|^2\mu_{X_N,\Xi_N}(dxd\xi)+\iint_{x\neq y}\tfrac{(\mu_{X_N}-\rho)(dx)(\mu_{X_N}-\rho)(dy)}{|x-y|}\end{aligned}$$

Assume that $(X_N, \Xi_N)(t)$ is a solution of Newton's system, while (ρ, u) is a solution of the Euler-Poisson system

$$
\begin{aligned}
&\frac{d}{dt}\mathcal{E}[X_N(t), \Xi_N(t), \rho(t,\cdot), u(t,\cdot)] \\
&= \tfrac{2}{N}\sum_{j=1}^{N}(\xi_j(t) - u(t,x_j(t)))\cdot(\dot{\xi}_j(t) - (\partial_t + \dot{x}_j(t)\cdot\nabla_x)u(t,x_j(t))) \\
&\quad +2\iint \frac{\rho(t,x)\partial_t\rho(t,y)}{|x-y|} - \tfrac{2}{N^2}\sum_{j\neq k}\frac{\dot{x}_j(t)\cdot(x_j(t)-x_k(t))}{|x_j(t)-x_k(t)|^3} \\
&\quad -\tfrac{2}{N}\sum_{j=1}^{N}\int\frac{\partial_t\rho(t,x)}{|x-x_j(t)|}dx + \tfrac{2}{N}\sum_{j=1}^{N}\int\frac{\rho(t,x)\dot{x}_j(t)\cdot(x_j(t)-x)}{|x-x_j(t)|^3}dx
\end{aligned}
$$

Then we eliminate $\dot{\xi}_j(t)$ and $\dot{x}_j(t)$ by using the Newton equations, and $\partial_t\rho$ and $\partial_t u$ by using the Euler-Poisson system

Denoting $V_\rho(t,\cdot)=\rho(t,\cdot)\star\frac{1}{|\cdot|}$, the equality above becomes

$$\frac{d}{dt}\mathcal{E}[X_N(t),\Xi_N(t),\rho(t,\cdot),u(t,\cdot)]$$

$$=\tfrac{2}{N}\sum_{j=1}^{N}(\xi_j(t)-u(t,x_j(t)))\cdot(\tfrac{1}{N}\sum_{k\neq j}\tfrac{x_j(t)-x_k(t)}{|x_j(t)-x_k(t)|^3}+\nabla V_\rho(t,x_j(t)))$$

$$-\tfrac{2}{N}\sum_{j=1}^{N}\nabla_x u(t,x_j(t)):(\xi_j(t)-u(t,x_j(t)))^{\otimes 2}$$

$$+2\int\rho(t,x)u(t,x)\cdot\nabla_x V_\rho(t,x)dx-\tfrac{2}{N^2}\sum_{j\neq k}\frac{\xi_j(t)\cdot(x_j(t)-x_k(t))}{|x_j(t)-x_k(t)|^3}$$

$$+\tfrac{2}{N}\sum_{j=1}^{N}\frac{\rho u(t,x)\cdot(x-x_j(t))}{|x-x_j(t)|^3}-\tfrac{2}{N}\sum_{j=1}^{N}\xi_j(t)\cdot\nabla_x V_\rho(t,x_j(t))$$

This equality is easily transformed into

$$\frac{d}{dt}\mathcal{E}[X_N(t),\Xi_N(t),\rho(t,\cdot),u(t,\cdot)]$$

$$= -\tfrac{2}{N}\sum_{j=1}^{N}\nabla_x u(t,x_j(t))) : (\xi_j(t) - u(t,x_j(t)))^{\otimes 2}$$

$$+\iint_{x\neq y}\tfrac{(u(t,x)-u(t,y))\cdot(x-y)}{|x-y|^3}(\mu_{X_N(t)}(dx) - \rho(t,x)dx)$$

$$\times(\mu_{X_N(t)}(dy) - \rho(t,x)dy)$$

For each $\rho \in L^\infty(\mathbf{R}^3)$, each $u \in W^{1,\infty}(\mathbf{R}^3)^3$ and each $X_N \in \mathbf{R}^{3N}$, define

$$\begin{cases} F[X_N,\rho] := \displaystyle\iint_{x\neq y} \frac{(\mu_{X_N}-\rho)(dx)(\mu_{X_N}-\rho)(dy)}{|x-y|} \\ G[X_N,\rho,u] := -\displaystyle\iint_{x\neq y} \frac{(u(x)-u(y))\cdot(x-y)}{|x-y|^3}(\mu_{X_N}-\rho)(dx)(\mu_{X_N}-\rho)(dy) \end{cases}$$

There exists $C>2$ such that, for all $\rho \in L^\infty(\mathbf{R}^3)$, all $u \in W^{1,\infty}(\mathbf{R}^3)^3$ and a.e. $X_N \in \mathbf{R}^{3N}$

$$|G[X_N,\rho,u]| \le C\|\nabla u\|_{L^\infty} F_N[X_N,\rho] + \frac{C}{N^{1/3}}(1+\|\rho\|_{L^\infty})(1+\|u\|_{W^{1,\infty}})$$

Therefore

$$\begin{aligned}&\frac{d}{dt}\mathcal{E}[X_N(t),\Xi_N(t),\rho(t,\cdot),u(t,\cdot)]\\ \le\ & C\|\nabla u(t,\cdot)\|_{L^\infty}\mathcal{E}[X_N(t),\Xi_N(t),\rho(t,\cdot),u(t,\cdot)]\\ &+\frac{C}{N^{1/3}}(1+\|\rho(t,\cdot)\|_{L^\infty})(1+\|u(t,\cdot)\|_{W^{1,\infty}})\end{aligned}$$

and one concludes by Gronwall's lemma that

$$\begin{aligned}&\mathcal{E}[X_N(t),\Xi_N(t),\rho(t,\cdot),u(t,\cdot)]\\ \le\ & e^{CLt}F[X_N^{in},\rho_N^{in}]+(1+\|\rho\|_{L^\infty})(1+\|u\|_{W^{1,\infty}})\frac{C(e^{CLt}-1)}{LN^{1/3}}\end{aligned}$$

with the control of the norms

$$\|\rho\|_{L^\infty([0,T]\times\mathbf{R}^3)}\,,\qquad\text{and}\qquad\|u(t,\cdot)\|_{W^{1,\infty}([0,T]\times\mathbf{R}^3)}$$

given by the existence theorem for the Euler-Poisson system on $[0,T]$.

Start from the decomposition (good exercise...)

$$\frac{1}{4\pi|x-y|} = \int_0^\infty dr \int_{\mathbf{R}^3} G_r(x-z)G_r(y-z)z$$

where

$$G_r(w) := \frac{1}{(2\pi r)^{3/2}} e^{-|w|^2/2r}$$

Then

$$F[X_N, \rho] \geq \int_\epsilon^\infty \|e^{r\Delta/2}(\mu_{X_N} - \rho)\|^2_{L^2(\mathbf{R}^3)} dr$$

Using the Banach-Alaoglu theorem and the uniqueness for the heat equation shows that

$$F[X_N, \rho] \to 0 \implies \mu_{X_N} \to \rho \text{ narrowly as } N \to \infty$$

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Kinetic Models — Lecture 2
Examples of Collisional Kinetic Models

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Summer School "From Kinetic Equations to Statistical Mechanics"
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The Boltzmann Equation

Unknown: (velocity) distribution function $F \equiv F(t, x, v) \geq 0$

Number of gas molecules at time t in an infinitesimal volume dx around x, with velocity belonging to an infinitesimal volume dv around v

$$F(t, x, v)dxdv$$

Boltzmann equation for F:

$$\underbrace{(\partial_t + v \cdot \nabla_x)F(t, x, v)}_{\text{free motion between collisions}} = \underbrace{\mathcal{B}(F, F)(t, x, v)}_{\substack{\text{gain/loss of molecules with velocity v} \\ \text{due to binary collisions}}}$$

The Boltzmann Collision Integral (Neutral Particles)

The Boltzmann collision integral acts on the variable v only

$$\mathcal{B}(F,F)(t,x,v) := \mathcal{B}(F(t,x,\cdot),F(t,x,\cdot))(v)$$

It is a bilinear integral operator whose expression is

$$\mathcal{B}(f,g)(v) := \iint_{\mathbf{R}^3\times\mathbf{S}^2} (f(v')g(v'_*) - f(v)g(v_*))b(v-v_*,\omega)dv_*d\omega$$

with "collision kernel" $b \equiv b(z,\omega) \geq 0$, where

$$\begin{cases} v' \equiv v'(v,v_*,\omega) := v - (v-v_*)\cdot\omega\,\omega \\ v'_* \equiv v'_*(v,v_*,\omega) := v_* + (v-v_*)\cdot\omega\,\omega \end{cases}$$

Notation: f' (resp. g'_* or g_*) designates $f(v')$ (resp. $g(v'_*)$ or $g(v_*)$)

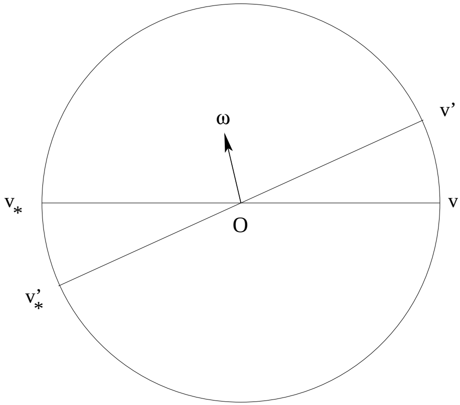

Figure: Collisions are assumed to be elastic, with geometry assuming the molecular radius is 0. The pre-collision relative velocity $v' - v'_*$ is mapped to the post-collision relative velocity $v - v_*$ by the reflection through the plane orthogonal to the unit vector ω.

Consider two colliding particles, radial repulsive interaction potential

$$U \equiv U(r) \in C^\infty((0,+\infty)) \text{ decreasing}$$
$$\lim_{r\to 0^+} U(r) = +\infty\,, \qquad \lim_{r\to+\infty} U(r) = 0$$

In the reference frame centered at one particle

let $v > 0$ be the **speed** of the moving particle **at infinity**
let $h := r(t)^2\dot\theta(t)/v$ = areal velocity$/v$ =: **impact parameter**
let $z_* > 0$ be the unique solution of $1 - z_*^2 - 4U(h/z_*)/v^2 = 0$, and

$$r_* = \frac{h}{z_*}\,, \quad \theta^* := \int_0^{z_*} \frac{dz}{\sqrt{1 - z^2 - 4U(h/z)/v^2}}$$

Trajectory in polar coordinates (with reflection through the angle θ^*)

$$\theta = \int_0^{h/r} \frac{dz}{\sqrt{1 - z^2 - 4U(h/z)/v^2}}\,, \quad r > r_*\,,\ 0 < \theta \le \theta_*$$

Trajectory of Colliding Particle

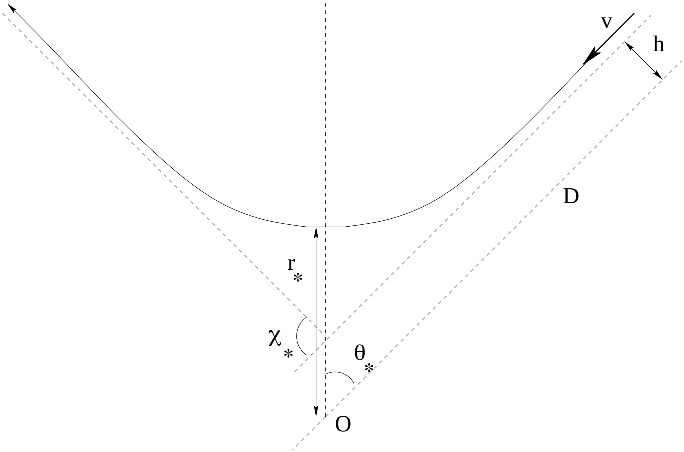

Figure: The impact parameter h, the apsis polar coordinates r_* and θ_*, the deflection angle $\chi_* = \pi - 2\theta_*$

For $v > 0$ fixed, the map $h \mapsto \chi_* \equiv \chi_*(v)$ is decreasing and s.t.

$$\lim_{h\to 0^+} \chi_*(h) = \pi_- \,, \qquad \lim_{h\to +\infty} \chi_*(h) = 0$$

Consider the map

$$\mathcal{S} : \mathbf{R}^2 \setminus \{0\} \ni (h, \phi) \mapsto (\chi_*(h), \phi) \in \mathbf{S}^2 \setminus \{N, S\}$$

where (h, ϕ) are the polar coordinates of a point in $\mathbf{R}^2 \setminus \{0\}$, while $(\chi_*(h), \phi)$ are the spherical coordinates of a point on $\mathbf{S}^2 \setminus \{N, S\}$

$$S(v,\chi) \underbrace{\sin\chi d\chi d\phi}_{\text{surface element on } \mathbf{S}^2} = \mathcal{S}\# \underbrace{h dh d\phi}_{\text{Lebesgue measure}}$$

Explicitly, the differential cross section is

$$S(v,\chi) := \left.\frac{h}{\sin\chi |\chi_*'(h)|}\right|_{\chi_*(h)=\chi}$$

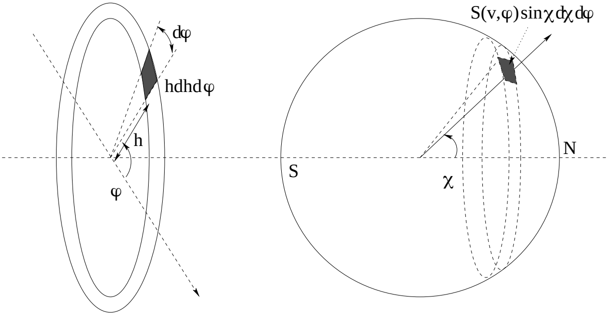

Figure: The collision cross section corresponding to the relative velocity v in the direction χ corresponding to the impact parameter h. In this figure, the moving particle approaches the origin O from the southern hemisphere, with the polar axis as the asymptote in the past.

Cross Section in Terms of the Unit Vector ω

In the Boltzmann equation, one uses $\theta = \frac{\pi-\chi}{2}$ instead of χ. Define

$$\tilde{\Sigma}(v, \cos\theta) := 4S(v, \pi - 2\theta)\cos\theta$$

Consider the C^∞-diffeomorphism

$$X : (0, \tfrac{\pi}{2}) \times (0, 2\pi) \ni (\theta, \phi) \mapsto (\chi = \pi - 2\theta, \phi) \in (0, \pi) \times (0, 2\pi)$$

With $V :=$ relative velocity (so that $v := |V|$)

$$X^{-1}\#S(|V|, \chi) \underbrace{\sin\chi d\chi d\phi}_{4\sin\theta\cos\theta d\theta d\phi} = \tilde{\Sigma}(|V|, \cos(\widehat{V,\omega}))\mathbf{1}_{V\cdot\omega>0} \underbrace{d\omega}_{\sin\theta d\theta d\phi}$$

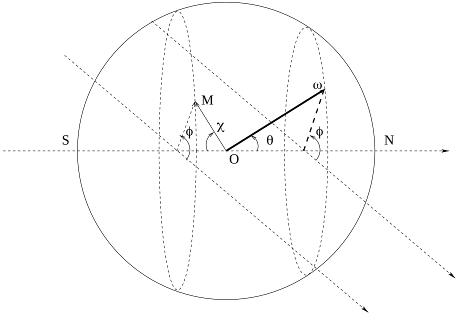

Figure: The vector ω, and the asymptotic direction $\vec{OM}$. The deflection angle is $\chi = \pi - 2\theta$, where θ is the colatitude of the unit vector ω. The moving particle approaches the origin O from the northern hemisphere, with the polar axis as the asymptote in the past. The asymptote in the future is the straight line OM.

Integrating Functions of $\mathcal{T}_\omega$

Define

$$\Sigma(v,\mu) := \tfrac{1}{2}\tilde{\Sigma}(v,\mu)$$

One easily checks on the definition of the collision map $\mathcal{T}_\omega$ that

$$\mathcal{T}_\omega = \mathcal{T}_{-\omega} \quad \text{for each } \omega \in \mathbf{S}^2$$

Thus, for each continuous $g : \mathbf{R}^3 \times \mathbf{R}^3 \to [0,+\infty)$, one has

$$\int_{V\cdot\omega>0} g(\mathcal{T}_\omega(v,v_*))\tilde{\Sigma}(|v-v_*|,\cos(\widehat{v-v_*,\omega}))d\omega$$
$$= \int_{\mathbf{S}^2} g(\mathcal{T}_\omega(v,v_*))\Sigma(|v-v_*|,|\cos(\widehat{v-v_*,\omega})|)d\omega$$

The Boltzmann collision kernel b is given by the formula

$$b(V,\omega) = |V|\Sigma(|V|,|\cos(\widehat{V,\omega})|)$$

Examples of Cross Sections

•If molecules=hard spheres of diameter $d_0 > 0$, the cross-section is

$$\Sigma(|V|, |\cos\theta|) = \tfrac{1}{2} d_0^2 |\cos\theta| \qquad \text{with } \theta = (\widehat{V, \omega})$$

•If molecules=points with inverse power law, repulsive potential

$$U(r) = k/r^s \,, \qquad \text{with } k, s > 0$$

one has

$$\begin{cases} \Sigma(|V|, |\cos\theta|) = \tfrac{1}{4}(2k)^{2/s} |V|^{-4/s} \beta(\theta)/\sin\theta & \text{with } \theta = (\widehat{V, \omega}) \\ \beta(\theta) = O(\theta) \text{ as } \theta \to 0^+ & \beta(\theta) = O\left((\tfrac{\pi}{2} - \theta)^{-1-\frac{2}{s}} \right) \text{ as } \theta \to \tfrac{\pi}{2}^- \end{cases}$$

•Maxwell molecules corresponding to $s = 4$
•Coulomb case $s = 1 \implies$ logarithmic divergence of Boltzmann's collision integral, has to be replaced with Landau's collision integral

Grad's Cutoff Assumption

With molecular interaction described by $U(r) = k/r^s$ with $k, s > 0$, the contributions of grazing collisions $\chi = 0$ or $\theta = \frac{\pi}{2}$ implies that

$$\int_{\mathbf{S}^2} b(|V|, \omega) d\omega = +\infty$$

H. Grad argued that grazing collisions are statistically unessential in neutral gases, and proposed to replace, for some $\theta_0 \in (0, \frac{\pi}{2})$

$$\varpi(\theta) := \tfrac{1}{4}(2k)^{2/s} \frac{\beta(\theta)}{\sin\theta} \quad \text{with} \quad \tilde{\varpi}(\theta)|\cos\theta|$$

where the function $\tilde{\varpi}$ satisfies

$$\tilde{\varpi} \in L^\infty([0, \pi])$$

Hard vs. Soft Cutoff Potentials

If the molecular interaction potential is $U(r) = k/r^s$ with $k, s > 0$

$$b(V,\omega) = |V|^{1-4/s}\tilde{\varpi}(\theta)|\cos\theta| \quad \text{with } \theta := (\widehat{V,\omega})$$

•Hard (cutoff) potentials: $s > 4$
•Pseudo-Maxwell molecules: $s = 4$
•Soft (cutoff) potentials: $1 < s < 4$

Setting $\gamma := 1 - \frac{4}{s}$, one has

$$0 < b(V,\omega) \le \|\tilde{\varpi}\|_{L^\infty}|V|^\gamma|\cos\theta| \quad \begin{cases} \gamma \in (0,+1) & \text{hard} \\ \gamma \in (-2,0) & \text{soft} \\ \gamma = 0 & \psi\text{-Maxwell} \end{cases}$$

•Hard spheres correspond to $\gamma = 1$ and $\tilde{\varpi}(\theta) \equiv 1$

$$b(V,\omega) = |V||\cos\theta| \quad \text{with } \theta := (\widehat{V,\omega})$$

Unknown: (velocity) distribution function $F \equiv F(t,x,v) \geq 0$

Landau equation for F:

$$(\partial_t + v \cdot \nabla_x)F(t,x,v) = \mathcal{C}(F)(t,x,v)$$

Landau collision integral:

$$\mathcal{C}(F)(t,x,v) = \operatorname{div}_v \int_{\mathbf{R}^3} a(v-w)(\nabla_v - \nabla_w)(F(t,x,v)F(t,x,w))dw$$

In the Coulomb case, a is given by the formula

$$a(z) := \nabla^2|z| = \frac{1}{|z|}(I - \Pi(z))\,, \qquad \Pi(z) := \left(\frac{z}{|z|}\right)^{\otimes 2}$$

Outline

In this lecture, we shall study the fundamental properties of the collision integrals $\mathcal{B}$ and $\mathcal{C}$

(a) weak formulation

(b) local conservation of mass, momentum and energy

(c) collision invariants

(d) Boltzmann's H Theorem and Maxwellians

Lemma 1
For each $\omega \in \mathbf{S}^2$, the map $\mathcal{T}_\omega : \mathbf{R}^3 \times \mathbf{R}^3 \to \mathbf{R}^3 \times \mathbf{R}^3$ defined by

$$\mathcal{T}_\omega(v, v_*) := (v'(v, v_*, \omega), v'_*(v, v_*, \omega))$$

satisfies the following properties

(a) one has $\mathcal{T}_{-\omega} = \mathcal{T}_\omega$ for each $\omega \in \mathbf{S}^2$, and, for each $v, v_* \in \mathbf{R}^3$

$$\mathcal{T}_\omega(v_*, v) = (v'_*(v, v_*, \omega), v'(v, v_*, \omega))$$

(b) the map $\mathcal{T}_\omega$ is an orthogonal symmetry on $\mathbf{R}^3 \times \mathbf{R}^3$; one has

$$\mathcal{T}_\omega^* = \mathcal{T}_\omega = \mathcal{T}_\omega^{-1} \quad \text{and} \quad \det(\mathcal{T}_\omega) = -1$$

For each $v, v_* \in \mathbf{R}^3$ and $\omega \in \mathbf{S}^2$, one has

$$\begin{aligned}\mathcal{T}_\omega(v_*, v) =&(v_* - (v_* - v)\cdot\omega\omega, v + (v_* - v)\cdot\omega\omega)\\ =&(v_* + (v - v_*)\cdot\omega\omega, v - (v - v_*)\cdot\omega\omega)\\ =&(v'_*(v, v_*, \omega), v'(v, v_*, \omega))\end{aligned}$$

Denoting $P_\omega z = z\cdot\omega\omega$, so that $P_\omega^* = P_\omega = P_\omega^2$, one has

$$\mathcal{T}_\omega = \begin{pmatrix} I - P_\omega & P_\omega \\ P_\omega & I - P_\omega \end{pmatrix} = \mathcal{T}_\omega^* = \mathcal{T}_\omega^{-1}$$

and, by the Schur complement

$$\det \mathcal{T}_\omega = \det((I - P_\omega)^2 - P_\omega^2) = \det(I - 2P_\omega) = -1$$

Collision Kernel

It is of the form

$$b(z,\omega) = |z| \underbrace{\Sigma(|z|, |\cos(\widehat{z,\omega})|)}_{\text{differential cross-section}}$$

Important special cases

(a) Hard spheres of radius $r > 0$:

$$b(z,\omega) = 2r^2 |z \cdot \omega|$$

(b) Maxwell molecules

$$b(z,\omega) = \mathbf{b}(|\cos(\widehat{z,\omega})|)$$

Notation For $p \geq 1$ and $m \in \mathbf{R}$, set $L^p_m(\mathbf{R}^d) := L^p(\mathbf{R}^3; (1+|v|^2)^{\frac{m}{2}} dv)$

Theorem 1

Assume that $0 \leq b(z,\omega) \leq C_b(1+|z|^2)$ for a.e. $(z,\omega) \in \mathbf{R}^3 \times \mathbf{S}^2$.

(1) For $m \geq 0$ and $f \in L^1_{2m+2}(\mathbf{R}^3)$, one has $\mathcal{B}(f,f) \in L^1_{2m}(\mathbf{R}^3)$, with

$$\int_{\mathbf{R}^3} (1+|v|^2)^m |\mathcal{B}(f,f)(v)| dv \leq 4^{m+2} \pi C_b \left(\int_{\mathbf{R}^3} (1+|v|^2)^{m+1} f(v) dv \right)^2$$

(2) For each ϕ s.t. $v \mapsto \frac{|\phi(v)|}{(1+|v|^2)^m}$ belongs to $L^\infty(\mathbf{R}^3)$, one has

$$\int_{\mathbf{R}^3} \mathcal{B}(f,f)(v)\phi(v) dv$$
$$= \iiint_{\mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{S}^2} (f' f'_* - f f_*) \frac{\phi + \phi_* - \phi' - \phi'_*}{4} b(v - v_*, \omega) dv dv_* d\omega$$

Proof of Theorem 1

Step 1: Write $\mathcal{B} = \mathcal{B}_+ - \mathcal{B}_-$ with

$$\mathcal{B}_+(f,f)(v) := \iint_{\mathbf{R}^3\times\mathbf{S}^2} f(v')f(v'_*)b(v-v_*,\omega)dv_*d\omega$$

$$\begin{aligned}\mathcal{B}_-(f,f)(v) :=& f(v)\iint_{\mathbf{R}^3\times\mathbf{S}^2} f(v_*)b(v-v_*,\omega)dv_*d\omega \\ =& f(v)(f\star\overline{b})(v) \quad \text{with } \overline{b}(z) = \int_{\mathbf{S}^2} b(z,\omega)d\omega\end{aligned}$$

Since $0 \le b(z,\omega) \le C_b(1+|z|^2)$, and $|v-v_*|^2 \le 2|v|^2 + 2|v_*|^2$

$$\begin{aligned}\mathcal{B}_-(f,f)(v) &\le C_b \iint_{\mathbf{R}^3\times\mathbf{S}^2} f(v)f(v_*)(1+2|v|^2+2|v_*|^2)dv_*d\omega \\ &\le 16C_b\pi(1+|v|^2)f(v)\int_{\mathbf{R}^3}(1+|v_*|^2)f(v_*)dv_*\end{aligned}$$

Step 2: On the other hand, by Tonelli's theorem

$$
\begin{aligned}
&\int_{\mathbf{R}^3}(1+|v|^2)^m\mathcal{B}_+(f,f)(v)dv\\
&\le C_b\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} f^{\otimes 2}(\mathcal{T}_\omega(v,v_*))(1+2|v|^2+2|v_*|^2)^{m+1}dvdv_*d\omega\\
&= C_b\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} f^{\otimes 2}(v,v_*)(1+2|v|^2+2|v_*|^2)^{m+1}dvdv_*d\omega\\
&\le 4^{m+2}\pi C_b\left(\int_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} f(v)(1+|v|^2)^{m+1}dv\right)^2
\end{aligned}
$$

The equality above follows from the fact that $\mathcal{T}_\omega$ is an orthogonal transformation of $\mathbf{R}^3\times\mathbf{R}^3$, and therefore leaves invariant the measure

$$(1+2|v|^2+2|v_*|^2)dvdv_*$$

Step 3: Property (a) of $\mathcal{T}_\omega$ and the structure of the collision kernel imply that

$$(f^{\otimes 2} \circ \mathcal{T}_\omega - f^{\otimes 2})(v, v_*) = (f^{\otimes 2} \circ \mathcal{T}_\omega - f^{\otimes 2})(v_*, v)$$
$$b(v - v_*, \omega) = b(v_* - v, \omega)$$

One has $(f'f'_* - ff_*)\phi \in L^1(\mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{S}^2, b(v - v_*, \omega)dvdv_*d\omega)$ by Steps 1 and 2 and Fubini's theorem, so that

$$J = \int_{\mathbf{R}^3} \phi(v)\mathcal{B}(f,f)(v)dv$$

$$= \iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} (f^{\otimes 2} \circ \mathcal{T}_\omega - f^{\otimes 2})(v, v_*) \frac{\phi(v)+\phi(v_*)}{2} b(v - v_*, \omega)dvdv_*d\omega$$

Step 4: Property (b) of $\mathcal{T}_\omega$ and the structure of the collision kernel imply that

$$\begin{cases}(f^{\otimes 2} - f^{\otimes 2} \circ \mathcal{T}_\omega) = (f^{\otimes 2} \circ \mathcal{T}_\omega^2 - f^{\otimes 2} \circ \mathcal{T}_\omega) \\ b(v'(v, v_*, \omega) - v'_*(v, v, \omega), \omega) = b(v - v_*, \omega)\end{cases}$$

Since $|\det \mathcal{T}_\omega| = 1$, the change of variables formula implies that

$$\begin{aligned} J &= \iiint_{\mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{S}^2} (f^{\otimes 2} \circ \mathcal{T}_\omega - f^{\otimes 2}) \frac{\phi(v)+\phi(v_*)}{2} b(v - v_*, \omega) dv dv_* d\omega \\ &= \iiint_{\mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{S}^2} (f^{\otimes 2} - f^{\otimes 2} \circ \mathcal{T}_\omega) \frac{\phi(v')+\phi(v'_*)}{2} b(v - v_*, \omega) dv dv_* d\omega \\ &= \iiint_{\mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{S}^2} (f' f'_* - f f_*) \frac{\phi+\phi_*-\phi'-\phi'_*}{4} b(v - v_*, \omega) dv dv_* d\omega \\ &= \iiint_{\mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{S}^2} f f_* \frac{\phi'+\phi'_*-\phi-\phi_*}{2} b(v - v_*, \omega) dv dv_* d\omega \end{aligned}$$

qed

Corollary 1

Assume that $0 \le b(z,\omega) \le C_b(1+|z|^2)$ for a.e. $(z,\omega) \in \mathbf{R}^3 \times \mathbf{S}^2$.

For each $f \in L^1_4(\mathbf{R}^3)$, and for $j = 1,2,3$, one has

$$\underbrace{\int_{\mathbf{R}^3} \mathcal{B}(f,f)(v)dv}_{\text{mass}} = \underbrace{\int_{\mathbf{R}^3} \mathcal{B}(f,f)(v)v_j dv}_{\text{momentum}} = \underbrace{\int_{\mathbf{R}^3} \mathcal{B}(f,f)(v)|v|^2 dv}_{\text{energy}} = 0$$

Check that, if $\phi \equiv 1$, or v_j for $j = 1, 2, 3$, or $|v|^2$, then

$$\phi(v) + \phi(v_*) = \phi(v') + \phi(v'_*)$$

for all $v, v_* \in \mathbf{R}^3$ and $\omega \in \mathbf{S}^2$. This is obvious for $\phi \equiv 1$.

Next, observe that

$$v' + v'_* = v - (v - v_*) \cdot \omega\omega + v_* + (v - v_*) \cdot \omega\omega = v + v_* \, .$$

Finally

$$\begin{aligned} |v'|^2 + |v'_*|^2 = & |v|^2 + |(v - v_*) \cdot \omega|^2 - 2(v - v_*) \cdot \omega \, \omega \cdot v \\ & + |v_*|^2 + |(v - v_*) \cdot \omega|^2 + 2(v - v_*) \cdot \omega \, \omega \cdot v_* = |v|^2 + |v_*|^2 \end{aligned}$$

qed

Exercise 1 For all $f, \phi \in \mathcal{S}(\mathbf{R}^3)$, prove that

$$\int_{\mathbf{R}^3} \phi(v)\mathcal{C}(f)(v)dv$$

$$= -\tfrac{1}{2} \iint_{\mathbf{R}^6} (\nabla\phi(v) - \nabla\phi(w)|a(v-w)(\nabla_v - \nabla_w)(f(v)f(w))dvdw$$

Exercise 2 Prove that, for all $f \in \mathcal{S}(\mathbf{R}^3)$ and all $j = 1, 2, 3$

$$\int_{\mathbf{R}^3} \mathcal{C}(f)(v)dv = \int_{\mathbf{R}^3} v_j\mathcal{C}(f)(v)dv = \int_{\mathbf{R}^3} |v|^2\mathcal{C}(f)(v)dv = 0$$

Definition

A collision invariant is a function $\phi:\ \mathbf{R}^3 \to \mathbf{R}$ such that

$$\phi(v'(v, v_*, \omega)) + \phi(v'_*(v, v_*, \omega)) = \phi(v) + \phi(v_*)$$

for a.e. $(v, v_*, \omega) \in \mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{S}^2$.

Theorem 2

Let $\phi \in C^2(\mathbf{R}^3)$. The function ϕ is a collision invariant iff there exist $\alpha, \gamma \in \mathbf{R}$ and $\beta \in \mathbf{R}^3$ such that

$$\phi(v) = \gamma|v|^2 + \beta \cdot v + \alpha \quad \text{for all } v \in \mathbf{R}^3$$

Proof of Theorem 2

Since

$$\phi(v'(v,v_*,\omega))+\phi(v'_*(v,v_*,\omega))=\phi(v)+\phi(v_*)$$

one has

$$\begin{aligned}0&=D_\omega(\phi(v'(v,v_*,\omega))+\phi(v'_*(v,v_*,\omega)))\cdot\zeta\\&=-((\nabla\phi(v'(v,v_*,\omega))-\nabla\phi(v'_*(v,v_*,\omega)))\cdot\omega)((v-v_*)\cdot\zeta)\\&\quad-((\nabla\phi(v'(v,v_*,\omega))-\nabla\phi(v'_*(v,v_*,\omega)))\cdot\zeta)((v-v_*)\cdot\omega)\end{aligned}$$

Specializing this to $\omega=\frac{v-v_*}{|v-v_*|}$ shows that $(v',v'_*)=(v_*,v)$, so that

$$\zeta\perp v-v_*\implies(\nabla\phi(v_*)-\nabla\phi(v))\cdot\zeta=0$$

so that

$$(v-v_*)\times(\nabla\phi(v_*)-\nabla\phi(v))=0$$

Differentiating in v in the direction u implies that

$$u \times (\nabla\phi(v_*) - \nabla\phi(v)) - (v - v_*) \times (\nabla^2\phi(v) \cdot u) = 0$$

Setting $u = \text{const.}(v - v_*)$, the first term above disappears and hence

$$u \times (\nabla^2\phi(v) \cdot u) = 0$$

Therefore, any nonzero vector u is an eigenvector of $\nabla^2\phi(v)$, so that

$$\nabla^2\phi(v) = \lambda(v)I$$

Returning to the identity above for all u, v, v_*

$$u \times (\nabla\phi(v_*) - \nabla\phi(v)) - \lambda(v)(v - v_*) \times u = 0$$

Differentiating both sides of this equality in v_* in the direction $\overline{u}$ shows that

$$u \times (\nabla^2\phi(v_*) \cdot \overline{u}) + \lambda(v)\overline{u} \times u = (\lambda(v_*) - \lambda(v))u \times \overline{u} = 0$$

Since this is true for all $v, v_*, u, \overline{u}$, we conclude that $\lambda(v) = \lambda(v_*)$ in other words that λ is a constant.

From $\nabla^2\phi(v) = \lambda I$, we deduce that, for some constant scalar α and constant vector field β, one has

$$\phi(v) = \tfrac{1}{2}\lambda|v|^2 + \beta \cdot v + \alpha$$

qed

Exercise 3 Find all $\phi \in C^2(\mathbf{R}^3)$ such that

$$a(v-w)(\nabla\phi(v)-\nabla\phi(w))=0$$

where we recall that

$$a(z)=\nabla^2|z|=\frac{1}{|z|}(1-\Pi(z))\,,\qquad \Pi(z):=\left(\frac{z}{|z|}\right)^{\otimes 2}$$

For $\epsilon > 0$, let G_ϵ be the centered Gaussian with covariance matrix $\epsilon^2 I$

$$G_\epsilon(v) := \frac{1}{\epsilon^3} G\left(\frac{v}{\epsilon}\right) , \quad \text{where } G(v) := \frac{1}{(2\pi)^{3/2}} e^{-\frac{|v|^2}{2}} , \quad v \in \mathbf{R}^3$$

Corollary 2

Let $\phi \in L^1(\mathbf{R}^3; G_{0,\epsilon_0}(v)dv)$ for some $\epsilon_0 > 0$. Then ϕ is a collision invariant iff there exist $\alpha, \gamma \in \mathbf{R}$ and $\beta \in \mathbf{R}^3$ such that

$$\phi(v) = \gamma |v|^2 + \beta \cdot v + \alpha \quad \text{for a.e. } v \in \mathbf{R}^3$$

For $0 < \epsilon < \epsilon_0$, one has

$$G_\epsilon \leq \left(\frac{\epsilon_0}{\epsilon}\right)^3 G_{\epsilon_0}$$

For $0 < \epsilon < \epsilon_0$, the function $\phi_\epsilon = \phi \star G_\epsilon$ belongs to $C^\infty(\mathbf{R}^3)$. Then, for each $\omega \in \mathbf{S}^2$ and each $v, v_* \in \mathbf{R}^3$, one has

$$\begin{aligned} &\phi_\epsilon(v') + \phi_\epsilon(v'_*) \\ &= \iint_{\mathbf{R}^3 \times \mathbf{R}^3} \underbrace{(\phi(w) + \phi(w_*)) G_\epsilon(v' - w) G_\epsilon(v'_* - w_*)}_{=:\Phi(w,w_*)} dw dw_* \\ &= \iint_{\mathbf{R}^3 \times \mathbf{R}^3} \underbrace{(\phi(w') + \phi(w'_*))\, G_\epsilon(v' - w')\, G_\epsilon(v'_* - w'_*)}_{=:\Phi(\mathcal{T}_\omega(w,w_*))} dw dw_* \end{aligned}$$

since $\mathcal{T}_\omega$ leaves the Lebesgue measure of $\mathbf{R}^3 \times \mathbf{R}^3$ invariant.

Then, by linearity of $\mathcal{T}_\omega$

$$\begin{aligned} G_\epsilon(v'-w')\,G_\epsilon(v'_*-w'_*) &= G_\epsilon\otimes G_\epsilon\circ\mathcal{T}_\omega(v-w,v_*-w_*)\\ &= G_\epsilon\otimes G_\epsilon(v-w,v_*-w_*)\\ &= G_\epsilon(v-w)\,G_\epsilon(v_*-w_*)\end{aligned}$$

Hence

$$\begin{aligned} &\phi_\epsilon(v')+\phi_\epsilon(v'_*)\\ &= \iint_{\mathbf{R}^3\times\mathbf{R}^3}\underbrace{(\phi(w')+\phi(w'_*))}_{=\phi(w)+\phi(w_*)}\,G_\epsilon(v-w)\,G_\epsilon(v_*-w_*)dwdw_*\\ &= \iint_{\mathbf{R}^3\times\mathbf{R}^3}(\phi(w)+\phi(w_*))\,G_\epsilon(v-w)\,G_\epsilon(v_*-w_*)dwdw_*\\ &= \phi_\epsilon(v)+\phi_\epsilon(v_*)\end{aligned}$$

By Theorem 2, there exists $\alpha_\epsilon,\gamma_\epsilon\in\mathbf{R}$ and $\beta_\epsilon\in\mathbf{R}^3$ such that

$$\phi_\epsilon(v)=\gamma_\epsilon|v|^2+\beta_\epsilon\cdot v+\alpha_\epsilon\,,\quad v\in\mathbf{R}^3\,,\;\epsilon\in(0,\epsilon_0)$$

In the limit as $\epsilon \to 0$, one has

$$\phi_\epsilon = \phi \star G_\epsilon \to \phi \quad \text{in } L^1_{loc}(\mathbf{R}^3)$$

Hence

$$\alpha_\epsilon \to \alpha\,, \quad \beta_\epsilon \to \beta\,, \quad \text{and } \gamma_\epsilon \to \gamma$$

so that

$$\phi(v) = \gamma|v|^2 + \beta \cdot v + \alpha \quad \text{for a.e. } v \in \mathbf{R}^3$$

qed

Definition

For each $u \in \mathbf{R}^3$, each $\rho \geq 0$ and each $\theta > 0$

$$\mathcal{M}[\rho, u, \theta](v) := \frac{\rho}{(2\pi\theta)^{3/2}} e^{-\frac{|v-u|^2}{2\theta}}$$

Observation: $\ln \mathcal{M}[\rho, u, \theta]$ is a collision invariant

$$\begin{aligned}\ln \mathcal{M}[\rho, u, \theta](v) &= \ln \frac{\rho}{(2\pi\theta)^{3/2}} - \frac{1}{2\theta}|v-u|^2 \\ &= \underbrace{-\frac{1}{2\theta}}_{=\gamma}|v|^2 + \underbrace{\frac{1}{\theta}u}_{=\beta} \cdot v + \underbrace{\ln \frac{\rho}{(2\pi\theta)^{3/2}} - \frac{1}{2\theta}|u|^2}_{=\alpha}\end{aligned}$$

In particular, for each $v, v_* \in \mathbf{R}^3$ and each $\omega \in \mathbf{S}^2$, one has

$$\begin{aligned}\mathcal{M}[\rho, u, \theta](v'(v, v_*, \omega))\mathcal{M}[\rho, u, \theta](v'_*(v, v_*, \omega)) \\ = \mathcal{M}[\rho, u, \theta](v)\mathcal{M}[\rho, u, \theta](v_*)\end{aligned}$$

Theorem 3
Assume that $0<b(z,\omega)\le C_b(1+|z|^2)$ for a.e. $(z,\omega)\in\mathbf{R}^3\times\mathbf{S}^2$.
Let $f\in L^1_{2m+2}(\mathbf{R}^3)$ satisfy

$$f(v)>0 \quad \text{and} \quad |\ln f(v)|\le C(1+|v|^2)^m \quad \text{for a.e. } v\in\mathbf{R}^3$$

Then

$$\int_{\mathbf{R}^3}\mathcal{B}(f,f)(v)\ln f(v)dv$$

$$=-\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2}\mathcal{P}[f](v,v_*,\omega)b(v-v_*,\omega)dvdv_*d\omega\le 0$$

where

$$\mathcal{P}[f](v,v_*,\omega):=\tfrac{1}{4}(f(v')f(v'_*)-f(v)f(v_*))\ln\left(\frac{f(v')fv'_*)}{f(v)f(v_*)}\right)$$

Theorem 3 (cont'd)
Moreover

$$\int_{\mathbf{R}^3} \mathcal{B}(f,f)(v) \ln f(v) dv = 0 \iff \mathcal{B}(f,f)(v) = 0 \text{ for a.e. } v \in \mathbf{R}^3$$

$$\iff \text{ there exists } \rho, \theta > 0 \text{ and } u \in \mathbf{R}^3 \text{ s.t. } f = \mathcal{M}[\rho, u, \theta] \text{ a.e.}$$

Applying Theorem 1 with $\phi = \ln f$ shows that

$$\int_{\mathbf{R}^3} \mathcal{B}(f,f)(v) \ln f(v) dv$$
$$= -\iiint_{\mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{S}^2} \mathcal{P}[f](v, v_*, \omega) b(v - v_*, \omega) dv dv_* d\omega$$

Since ln is an increasing function

$$\mathcal{P}[f](v, v_*, \omega) \geq 0 \quad \text{for a.e. } (v, v_*, \omega) \in \mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{S}^2$$

Since $b > 0$ a.e. on $\mathbf{R}^3 \times \mathbf{S}^2$, the inequality in the H Theorem holds. Besides

$$\int_{\mathbf{R}^3} \mathcal{B}(f,f)(v) \ln f(v) dv = 0 \iff \mathcal{P}[f] = 0 \text{ a.e. on } \mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{S}^2$$
$$\iff \mathcal{B}(f,f) = 0 \text{ a.e. on } \mathbf{R}^3$$

Moreover

$$\mathcal{P}[f] = 0 \text{ a.e. on } \mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{S}^2$$
$$\iff \ln f \text{ is a collision invariant}$$

Since $|\ln f(v)| \leq C(1+|v|^2)^m$, one has $\ln f \in L^2(\mathbf{R}^3; G_\epsilon(v)dv)$ for each $\epsilon > 0$. By Corollary 2, there exists $\alpha, \gamma \in \mathbf{R}$ and $\beta \in \mathbf{R}^3$ s.t.

$$\ln f(v) = \gamma|v|^2 + \beta \cdot v + \alpha \quad \text{for a.e. } v \in \mathbf{R}^3$$

Hence $f = \mathcal{M}[\rho, u, \theta]$ a.e. on $\mathbf{R}^3$, with

$$\theta := -\frac{1}{2\gamma}, \quad u := -\frac{1}{2\theta}\beta, \quad \text{and } \rho := \left(\frac{\pi}{|\gamma|}\right)^{3/2} e^{\alpha - |\beta|^2/4\gamma}$$

In particular, one must have $\gamma < 0$.

Exercise 4 Let $f \in \mathcal{S}(\mathbf{R}^3)$ be such that $f > 0$ and $\ln f$ has polynomial growth at infinity. Then

$$0 \geq \int_{\mathbf{R}^3} \ln f(v) \mathcal{C}(f)(v) dv$$

$$= -\frac{1}{2} \iint_{\mathbf{R}^6} \operatorname{trace}(a(v-w)(\nabla \ln f(v) - \nabla \ln f(w))^{\otimes 2}) f(v) f(w) dv dw$$

Prove that

$$\int_{\mathbf{R}^3} \ln f(v) \mathcal{C}(f)(v) dv = 0 \iff \mathcal{C}(f) = 0 \iff f(v) \text{ is a Maxwellian}$$

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Kinetic Models — Lecture 3
The Regularity Problem for the Landau Equation

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Summer School "From Kinetic Equations to Statistical Mechanics"
Saint Jean de Monts, June 28th – July 2nd 2021

(Space Homogeneous) Landau Equation

Landau equation with unknown $f \equiv f(t,v) \geq 0$:

$$\partial_t f(t,v) = \underbrace{\mathrm{div}_v \int_{\mathbf{R}^3} a(v-w)(\nabla_v - \nabla_w)(f(t,v)f(t,w))dw}_{=:\Lambda(f)(t,v)}, \quad v \in \mathbf{R}^3$$

with the notation:

$$a(z) := \tfrac{1}{8\pi}\nabla^2|z| = \tfrac{1}{8\pi|z|}\Pi(z), \quad \Pi(z) := I - \left(\tfrac{z}{|z|}\right)^{\otimes 2}$$

Nonconservative form

$$\partial_t f(t,v) = (a_{ij} \star_v f(t,v))\, \partial_{v_i}\partial_{v_j} f(t,v) + f(t,v)^2$$

Open question Global existence of classical solutions or finite-time blow-up for the Cauchy problem with $f\big|_{t=0} = f_{in}$?

Conserved Quantities

Assume $f \equiv f(t,v) > 0$ smooth+rapidly decaying as $|v| \to \infty$, and let $\phi \equiv \phi(v)$ be smooth with at most polynomial growth as $|v| \to \infty$

$$\frac{d}{dt}\int_{\mathbf{R}^3}\phi(v)f(t,v)dv = -\iint_{\mathbf{R}^6}\mathcal{D}[\phi,f](t,v,w)f(t,v)f(t,w)dvdw$$

with $\mathcal{D}[\phi,f] \equiv \frac{1}{2}a_{ij}(v-w)(\partial_{v_i}\phi(v)-\partial_{w_i}\phi(w))\left(\frac{\partial_{v_j}f(t,v)}{f(t,v)}-\frac{\partial_{w_j}f(t,w)}{f(t,w)}\right)$

(a) Mass and momentum conservation laws: with $\phi \equiv 1$ or $\phi \equiv v_j$

$$\frac{d}{dt}\int_{\mathbf{R}^3}f(t,v)dv = \frac{d}{dt}\int_{\mathbf{R}^3}v_jf(t,v)dv = 0\,,\quad j=1,2,3$$

(b) Energy conservation law: with $\phi = \frac{1}{2}|v|^2$, one has

$$\nabla\phi(v)-\nabla\phi(w) = v-w \implies a(v-w)(\nabla\phi(v)-\nabla\phi(w)) = 0$$

$$\implies \frac{d}{dt}\int_{\mathbf{R}^3}\tfrac{1}{2}|v|^2f(t,v)dv = 0$$

H Theorem

(1) Assume $f \equiv f(t,v) > 0$ smooth+rapidly decaying as $|v| \to \infty$ s.t. $\ln f$ has at most polynomial growth as $|v| \to \infty$; then

$$\frac{d}{dt}\int_{\mathbf{R}^3} f \ln f(t,v)dv = -\iint_{\mathbf{R}^6} \mathcal{D}[\ln f, f](t,v,w)f(t,v)f(t,w)dvdw$$

$$= -\frac{1}{2}\iint_{\mathbf{R}^6} \operatorname{tr}\left(a(v-w)\left(\frac{\nabla_v f(t,v)}{f(t,v)} - \frac{\nabla_v f(t,v)}{f(t,v)}\right)^{\otimes 2}\right) f(t,v)f(t,w)dvdw \le 0$$

(2) One has the equivalences

$$\Lambda(f)(t,v) = 0 \iff \int_{\mathbf{R}^3} \Lambda(f)\ln f(t,v)dv = 0$$

$$\iff f(t,v) = \mathcal{M}[\rho,u,\theta](v) := \frac{\rho}{(2\pi\theta)^{3/2}}e^{-|v-u|^2/2\theta}$$

for some $\rho, \theta > 0$ and $u \in \mathbf{R}^3$

Indeed

$$\mathcal{D}[\ln f, f] = 0 \iff \frac{\nabla_v f(t,v)}{f(t,v)} = av + b$$

(1) Obviously, if $f(t,v) = f(t)$, the nonconservative form of the Landau equation reduces to the Riccati equation

$$\partial_t f = f^2 \implies \text{finite-time blow-up}$$

BUT $f(t) \in L^1(\mathbf{R}^3)$ only if $f(t) \equiv 0$...

(2) **Semilinear heat equation** there is finite-time blow-up for $u \geq 0$ solution of

$$\partial_t u = \Delta_x u + \alpha u^2$$

Hint Riccati inequality $\dot{L}(t) \geq -\lambda_0 L(t) + \alpha L^2(t)$ satisfied by

$$L(t) := \frac{\int_B u(t,x)\phi(x)dx}{\int_B \phi(x)dx} \quad \text{with} \begin{cases} -\Delta\phi = \lambda_0\phi\,, & \phi > 0 \text{ on } B \\ \phi\big|_{\partial B} = 0 \end{cases}$$

Or Not Blow-Up?

The diffusion matrix $a_{ij} \star_v f$ in the Landau equation increases with $f \geq 0$. While the quadratic term f^2 promotes blow-up, any local concentration of mass in f will feed the smoothing effect of the diffusion term, and offset the effect of the quadratic nonlinearity.

$$\partial_t f(t,v) = a_{ij} \star_v f(t,v) \partial_{v_i} \partial_{v_j} f(t,v) + f(t,v)^2$$

"Isotropic Landau" global existence of radially symmetric nonincreasing soln [Gressman-Krieger-Strain 2012, Gualdani-Guillen 2016]

$$\partial_t u = ((-\Delta)^{-1} u)\Delta u + \alpha u^2$$

Conditional regularity $L^\infty_t L^p_k$ solns with $p > \frac{3}{2}$ and $k > 5$ are $L^\infty_{t,v}$ ([Silvestre 2017], radial solns [Gualdani-Guillen 2016])

Elliptic Regularity with Bounded Coefficients

E. De Giorgi *Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari*. Mem. Accad. Sci. Torino Cl. Sci. Fis. Mat. Nat. (3) **3** (1957), 25–43

Let $u \equiv u(x) \in \mathbf{R}$ be a weak (variational) solution of

$$(L) \qquad -\partial_{x_i}(A_{ij}(x)\partial_{x_j}u(x)) = 0$$

Assume there exists constants $0 < m < M$ such that

$$(E) \qquad m|\xi|^2 \le A_{ij}(x)\xi_i\xi_j \le M|\xi|^2\,, \qquad x, \xi \in \mathbf{R}^d$$

•If A is continuous, zooming in near $x = 0$ and setting $u(x) = U(X)$ with $x = \epsilon X$ for $0 < \epsilon \ll 1$ leads to

$$-\partial_{X_i}(A_{ij}(\epsilon X)\partial_{X_j}U(X)) = 0 \quad \text{with } A(\epsilon X) \simeq A(0) = A(0)^T > 0$$

Then U (and therefore u) should behave like a harmonic function

•If A has, say a jump discontinuity at 0, zooming in will not make the coefficients $A_{ij}(x)$ nearly constant, hence the analogy above fails

Let $u \in H^1(\Omega)$ be a variational subsolution of (L) on Ω, i.e.

$$\int_\Omega (A(x)\nabla u(x)|\nabla v(x))dx \le 0 \quad \text{for all } v \in H^1_0(\Omega) \text{ s.t. } v \ge 0 \text{ on } \Omega$$

For all $x_0 \in \Omega$ and all $0 < r < R < \mathrm{dist}(x_0, \partial\Omega)$

$$\int_{B(x_0,r)} |\nabla u_+(x)|^2 dx \le \frac{M/m}{(R-r)^2} \int_{B(x_0,R)\setminus\overline{B(x_0,r)}} u_+(x)^2 dx$$

Proof of Caccioppoli's Inequality

$$
\begin{aligned}
0 \geq (A\nabla u|\nabla(\phi^2 u_+)) =&(A\nabla u|\phi\nabla(\phi u_+)) + (A\nabla u|\phi u_+\nabla\phi) \\
=&(\phi A\nabla u|\nabla(\phi u_+)) + (\phi A\nabla u|u_+\nabla\phi) \\
=&(A(\nabla(\phi u_+) - u_+\nabla\phi)|\nabla(\phi u_+)) \\
&+ (A(\nabla(\phi u_+) - u_+\nabla\phi)|u_+\nabla\phi) \\
=&(A\nabla(\phi u_+)|\nabla(\phi u_+)) - (u_+ A\nabla\phi)|u_+\nabla\phi)
\end{aligned}
$$

so that

$$
m\|\nabla(\phi u_+)\|^2_{L^2(\Omega)} \leq M\|u_+\nabla\phi\|^2_{L^2(\Omega)}
$$

Choose

$$
\phi(x) := \max\left(0, \min\left(1, \frac{R-|x|}{R-r}\right)\right)
$$

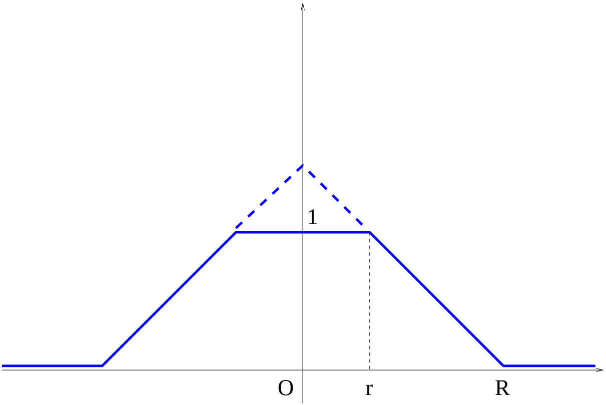

Figure: Section of the graph of the truncation function ϕ in Caccioppoli's inequality.

The De Giorgi Method (1957)

•In 1957, De Giorgi proved that, for some $\alpha \equiv \alpha[m, M] > 0$, the solutions u of (E) satisfy $u \in H^1_{loc} \implies u \in C^{0,\alpha}$.

This was the missing step in the solution of Hilbert's 19th problem (1900) — the **real-analyticity** of extremals of

$$H^1(\Omega) \ni u \mapsto \int_\Omega L(\nabla u(x))dx \in \mathbf{R}$$

where L is real-analytic with Hessian $\nabla^2 L$ satisfying condition (E)

De Giorgi's 1st Lemma For each $R > 0$, there exists $\eta[m, M, R]$ such that H^1 variational solutions u of (L) satisfy

$$\int_{|x|<R} |u(x)|^2 dx < \eta \implies u(x) \le 1 \text{ for } |x| \le R/2$$

Proof Let $R > r > 0$ and $\Lambda > \lambda > \frac{1}{2}$, set $2^* = \frac{2d}{d-2}$, by Hölder

$$\int_{|x|\le r} (u-\Lambda)_+^2(x)dx = \int_{|x|\le r} (u-\Lambda)_+^2(x) \underbrace{\mathbf{1}_{u(x)-\lambda\ge\Lambda-\lambda}}_{\text{useless? not quite}} dx$$

$$\le \left(\int_{|x|\le r} (u-\Lambda)_+^{2^*}(x)dx\right)^{2/2^*} |\{x : u(x)-\lambda \ge \Lambda-\lambda\}|^{2/d} dx$$

$$\le \underbrace{C_S^2 \int_{|x|\le r} \nabla(u-\Lambda)_+^2(x)dx}_{\text{Sobolev's embedding}} \underbrace{\left(\frac{\int_{|x|\le r}(u-\lambda)_+^2(x)dx}{(\Lambda-\lambda)^2}\right)^{2/d}}_{\text{Bienaymé-Chebyshev}}$$

$$\le \underbrace{C_S^2 \frac{\frac{M}{m}\int_{r\le|x|\le R}(u-\Lambda)_+^2(x)dx}{(R-r)^2}}_{\text{Caccioppoli's inequality}} \left(\frac{\int_{|x|\le r}(u-\lambda)_+^2(x)dx}{(\Lambda-\lambda)^2}\right)^{2/d}$$

$$\le C_S^2 \frac{M}{m} \frac{\left(\int_{|x|\le R}(u-\lambda)_+^2(x)dx\right)^{1+2/d}}{(R-r)^2(\Lambda-\lambda)^{4/d}}$$

Key observation NONLINEAR inequality for a LINEAR equation!

Choose $r_n := \frac{1}{2}R(1+2^{-n})$ and $\lambda_n = 1-2^{-n}$, and set

$$U_n^2 = \int_{|x|\le r_n} (u-\lambda_n)_+^2(x)dx$$

The chain of inequalities in the previous slides shows that

$$U_n \le \gamma \cdot 2^{n\beta} \cdot U_{n-1}^{\beta} \qquad \text{with } \beta := 1+\tfrac{2}{d} \text{ and } \gamma := \tfrac{2C_S}{R}\sqrt{\tfrac{M}{m}}$$

Setting $\eta := \gamma^{-d/2} \cdot 2^{(1+\frac{d}{2})^2}$, it is easy to see that, by Fatou's lemma

$$U_0 < \eta \implies \int_{|x|\le R/2} (u(x)-1)_+^2 dx \le \lim_{n\to\infty} U_n^2 = 0$$

Partial Regularity for the Landau Equation

M.P. Gualdani, F.G., C. Imbert and A. Vasseur
arXiv:1906.02841 [math.AP]
to appear in Ann. scient. École norm. sup.

In the context of the Landau equation, what replaces the elliptic variational principle used by De Giorgi is the H Theorem

Truncated H function

$$H_+(f|\kappa) := \int_{\mathbf{R}^3} \kappa h_+\left(\frac{f(v)}{\kappa}\right) dv$$

with

$$\begin{aligned} h_+(z) &:= z(\ln z)_+ - (z-1)_+ \\ &= (z\ln z - z + 1)\mathbf{1}_{z\geq 1} \end{aligned}$$

For each $\kappa > 0$

$$\frac{d}{dt}H_+(f(t,\cdot)|\kappa) \underbrace{+\int \frac{f(t,v)f(t,w)}{16\pi|v-w|}\left|\Pi(v-w)\left(\frac{\mathbf{1}_{f(t,v)>\kappa}\nabla_v f(t,v)}{f(t,v)} - \frac{\mathbf{1}_{f(t,w)>\kappa}\nabla_w f(t,w)}{f(t,w)}\right)\right|^2 dvdw}_{D_1}$$

$$= -\int a_{ij}(v-w)\partial_{v_i} f(t,v)\mathbf{1}_{f(t,v)\geq\kappa}\partial_{w_j} f(t,w)\mathbf{1}_{f(t,w)<\kappa}dvdw$$

$$= \int \underbrace{\partial_i\partial_j a_{ij}(v-w)}_{=\delta(v-w)}(f(t,v)-\kappa)_+(\kappa-(f(t,w)-\kappa)_-)dvdw$$

$$= \underbrace{\kappa\int(f(t,v)-\kappa)_+ dv}_{\substack{\text{depleted nonlinearity} \\ \text{weaker than } f(t,v)^2}}$$

Notation $\|g\|^p_{L^p_k} := \int (1+|v|^2)^{k/2}|g(v)|^p dv$ with $p \geq 1$ and $k \in \mathbf{R}$

H-solution $f \in C([0,T);\mathcal{D}'(\mathbf{R}^3)) \cap L^1((0,T);L^1_{-1}(\mathbf{R}^3))$ s.t. $f \geq 0$

$$\int_{\mathbf{R}^3} \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} f(t,v)dv = \int_{\mathbf{R}^3} \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} f_{in}(v)dv$$

$$\int_{\mathbf{R}^3} f(t,v)\ln f(t,v)dv \leq \int_{\mathbf{R}^3} f_{in}(v)\ln f_{in}(v)dv$$

for a.e. $t \geq 0$, and

$$\int_{\mathbf{R}^3} f_{in}(v)\phi(0,v)dv + \int_0^T \int_{\mathbf{R}^3} f(t,v)\partial_t\phi(t,v)dv$$
$$= \int_0^T \iint_{\mathbf{R}^6} \mathcal{D}[\phi,f](t,v,w)f(t,v)f(t,w)dvdwdt$$

Definition Let $\mathcal{N} \subset \mathbf{R}$ with $|\mathcal{N}| = 0$, let $q \geq 1$ and let $C_E > 0$. An $(\mathcal{N}, q, C_E)$-suitable solution of the Landau equation on the set $[0, T) \times \mathbf{R}^3$ is an H-solution s.t.

$$H_+(f(t_2, \cdot)|\kappa) + C_E \int_{t_1}^{t_2} \left\| \mathbf{1}_{f(t,v)>\kappa} \nabla_v f(t,v)^{1/q} \right\|_{L^q(\mathbf{R}^3)}^2 dt \\ \leq H_+(f(t_1, \cdot)|\kappa) + 2\kappa \int_{t_1}^{t_2} \int_{\mathbf{R}^3} (f(t,v) - \kappa)_+ dvdt$$

for all $t_1 < t_2 \in [0, T) \setminus \mathcal{N}$ and $\kappa \geq 1$

Remark The NONLOCAL entropy production term

$$\int \mathcal{D}[\ln f, f](t,v,w) f(t,v) f(t,w) dw \text{ replaced with } \underbrace{\simeq |\nabla_v f(t,v)|^2}_{\textbf{LOCAL dissipation}}$$

Prop 1 For all $0 \leq f_{in} \in L^1(\mathbf{R}^3)$ s.t.

$$\int_{\mathbf{R}^3} (1 + |v|^k + |\ln f_{in}(v)|) f_{in}(v) dv < \infty \qquad \text{for some } k > 3$$

there exists an $(\mathcal{N}, q, C_E)$-suitable solution f of the Landau equation on $[0, T]$ with initial data f_{in} and

$$C_E \equiv C_E[T, q, f_{in}] > 0\,, \qquad q := \frac{2k}{k+3}$$

The proof uses the same approximation scheme used by Villani to prove the global existence of H solutions + the Desvillettes theorem

Desvillettes Theorem

Thm For each $0 \leq f \in L^1_2(\mathbf{R}^3)$ s.t. $f \ln f \in L^1(\mathbf{R}^3)$

$$\int_{\mathbf{R}^3} \frac{|\nabla\sqrt{f(v)}|^2 dv}{(1+|v|^2)^{3/2}} \leq C_D + C_D \int_{\mathbf{R}^6} \frac{|\Pi(v-w)(\nabla_v - \nabla_w)\sqrt{f(v)f(w)}|^2}{|v-w|} dv dw$$

with

$$C_D \equiv C_D \left[\int_{\mathbf{R}^3} (1, v, |v|^2, |\ln f(v)|) f(v) dv\right] > 0$$

Corollary Let $0 \leq f_{in} \in L^1_k(\mathbf{R}^3))$ with $k > 2$ s.t. $f_{in}|\ln f_{in}| \in L^1(\mathbf{R}^3)$.

$$f \text{ H-solution s.t. } f\Big|_{t=0} = f_{in} \implies f \in L^\infty(0,T; L^1_k(\mathbf{R}^3))$$

Definition A regular time of f, suitable solution on $I \subset (0, +\infty)$, is a time $\tau \in I$ s.t. $f \in L^\infty((\tau - \epsilon, \tau) \times \mathbf{R}^3)$ for some $\epsilon \in (0, \tau)$.
The set of singular (i.e. nonregular) times of f on I is denoted $\mathbf{S}[f, I]$.

Main Thm Let f be a suitable solution to the Landau equation on $[0, T) \times \mathbf{R}^3$ for all $T > 0$, with initial data f_{in} satisfying

$$\int_{\mathbf{R}^3} (1 + |v|^k + |\ln f_{in}(v)|) f_{in}(v) dv < \infty \quad \text{for all } k > 3$$

Then

$$\text{Hausdorff dim } \mathbf{S}[f, (0, +\infty)] \leq \tfrac{1}{2}$$

Remark In 1934, Leray proved that the sets of singular times of his global "turbulent" solutions of the Navier-Stokes equation in $\mathbf{R}^3$ have Hausdorff dimension $\leq \frac{1}{2}$.

Hausdorff measure For each $S \subset \mathbf{R}$, each $d \geq 0$ and each $\delta > 0$

$$\mathcal{H}^d(S) := \sup_{\delta>0} \inf \left\{ \sum_{j\geq 1} \mathrm{diam}(U_j)^d, \quad S \subset \bigcup_{j\geq 1} U_j\,, \quad \mathrm{diam}(U_j) < \delta \right\}$$

This is a metric outer measure. All Borel sets in $\mathbf{R}$ are $\mathcal{H}^d$-measurable.

Hausdorff dimension For $S \subset \mathbf{R}$

$$\mathcal{H} - \dim(S) = \inf\{d \geq 0 \text{ s.t. } \mathcal{H}^d(S) = 0\}$$

The Cantor Middle Third Set

$$C := \{\sum_{n\geq 1} a_n 3^{-n}, \quad a_n \in \{0,2\}\} \implies \mathcal{H} - \dim(C) = \tfrac{\ln 2}{\ln 3}$$

The 1st De Giorgi Type Lemma

Prop 2 Let f be a $(\mathcal{N}, q, C_E)$–suitable solution to the Landau equation for $t \in [0,1]$ with $C_E > 0$ and $q \in (\frac{6}{5}, 2)$

Then there exists $\eta_0 \equiv \eta_0[q, C_E] > 0$ s.t.

$$\int_{1/8}^{1} H_+(f(t,\cdot)|\tfrac{1}{2})dt < \eta_0 \implies f(t,v) \le 2 \quad \text{a.e. on } [\tfrac{1}{2}, 1] \times \mathbf{R}^3$$

Prop 3 Let f be a $(\mathcal{N}, q, C_E)$-suitable solution to the Landau equation on $[0,1]$ with $q \in (\frac{4}{3}, 2)$. There exists $\eta_1 \equiv \eta_1[q, C_E] > 0$ and $\delta_1 \in (0,1)$ such that

$$\overline{\lim_{\epsilon \to 0^+}} \epsilon^{\gamma-3} \int_{1-\epsilon^\gamma}^{1} \left\| \mathbf{1}_{f(T,V)>\epsilon^{-\gamma}} \nabla_V f(T,V)^{\frac{1}{q}} \right\|_{L^q(\mathbf{R}^3)}^2 dT < \eta_1$$
$$\implies f \in L^\infty((1-\delta_1, 1) \times \mathbf{R}^3)$$

with $\gamma := \frac{5q-6}{2q-2}$.

Proof of Prop 3: (a) Scaling

•2-parameter group of invariance scaling transfo. for the Landau eq.:

$$f_{\lambda,\epsilon}(t,v) := \lambda f(\lambda t, \epsilon v)$$

•Let f be a $(\mathcal{N}, q, C_E)$-suitable solution on $[0,1]$, with $\lambda = \epsilon^\gamma$

$$H_+(f_{\lambda,\epsilon}(t,\cdot)|\epsilon^\gamma\kappa) = \epsilon^{\gamma-3}H_+(f(\epsilon^\gamma t,\cdot)|\epsilon^\gamma\kappa)$$

$$\int_{t_1}^{t_2}\int(f_{\lambda,\epsilon}(t,v)-\epsilon^\gamma\kappa)_+dvdt = \frac{1}{\epsilon^3}\int_{\epsilon^\gamma t_1}^{\epsilon^\gamma t_2}\int f(T,V)-\kappa)_+dVdT$$

while $\gamma := \frac{5q-6}{2q-2}$ implies that

$$\int_{t_1}^{t_2}\left(\int|\mathbf{1}_{f_{\lambda,\epsilon}\ge\epsilon^\gamma\kappa}\nabla_v f_{\lambda,\epsilon}^{\frac{1}{q}}(t,v)|^q dv\right)^{2/q}dt$$

$$= \epsilon^{\gamma-3}\int_{\epsilon^\gamma t_1}^{\epsilon^\gamma t_2}\left(\int|\mathbf{1}_{f\ge\kappa}\nabla_v f^{\frac{1}{q}}(T,V)|^q dV\right)^{2/q}dT$$

•Denoting $\mu(r) = \min(r, r^2)$, set

$$f_n(t,v) := \epsilon_n^\gamma f(1+\epsilon^\gamma(t-1), \epsilon v) \quad \text{with } \epsilon_n := 2^{-n}$$

$$F_n(t,v) := \mu((f_n(t,v)^{1/q}-1)_+)\,, \quad \int F_n(t,v)dv \le \epsilon_n^{\gamma-3}$$

•Observe that f_n is a $(\mathcal{N}_n, q, C_E)$-suitable solution of the Landau eq. on $[0,1]$ with

$$\mathcal{N}_n := \{t \ge 0 \text{ s.t. } 1+\epsilon_n^\gamma(t-1) \in \mathcal{N}\}$$

Key point: the constant C_E is **unchanged** by the scaling
•There exists N large enough so that

$$n \ge N \implies \int_0^1 \left(\int |\nabla_v F_n(t,v)|^q dv\right)^{2/q} dt$$

$$\le 4\epsilon_n^{\gamma-3} \int_{1-\epsilon_n^\gamma}^1 \left(\int |\mathbf{1}_{f \ge \epsilon_n^{-\gamma}} \nabla_V f(T,V)^{1/q}|^q dV\right)^{2/q} dT < 8\eta_1$$

Proof of Prop 3: (b) Iteration

•Use the Hölder inequality + Sobolev inequality as in the proof of Prop 2, isolating the term $\|\nabla_v F_{n+1}\|_{L^2_t L^q_v} = O(\eta_1)$ shows that

$$X_m := \text{ess-sup}_{\frac{1}{2}<t<1} \int F_{N+m}(t,v)^q dv$$

satisfies

$$X_{m+1} < \rho(\max(1, X_m)^\alpha + \max(1, X_{m-1})^\alpha)\,, \quad X_0, X_1 \le M$$
$$\text{with } \alpha := q/3\,, \qquad \rho := D(q)\eta_1^{q/2}\,, \qquad M := 2^{(N+3)(3-\gamma)}$$

•With η_1 small so that $\rho < \frac{1}{2}$, an easy induction shows that

$$X_{2m}, X_{2m+1} \le \max\left(2\rho, (2\rho)^{\frac{1-\alpha^m}{1-\alpha}} M^{\alpha^m}\right) \implies X_{m_0} < 2D(q)\eta_1^{\frac{q}{2}} \ll 1$$
$$\implies f_{N+m_0+3} \text{ satisfies the assumptions of Prop 2}$$

•By Prop 1, the initial data f_{in} launches a $(\mathcal{N}, q, C_E)$ suitable solution with a constant $C_E[T, f_{in}, q]$ for each $q \in (1, 2)$

•If $\tau \in \mathbf{S}[f, [1, 2]]$, apply Prop 3 to $f_\tau(t, v) := f(t + \tau - 1, v)$; for each $q \in (\frac{4}{3}, 2)$, there exists $\epsilon(\tau) \in (0, \frac{1}{2})$ s.t.

$$\int_{\tau-\epsilon(\tau)^\gamma}^{\tau} \left(\int |\nabla_v (f(t, v)^{1/q} - 1)_+|^q dv \right)^{2/q} dt \geq \tfrac{1}{2}\eta_1 \epsilon(\tau)^{3-\gamma}$$

•By Vitali's covering thm, there is a sequence $\tau_j \in \mathbf{S}[f, [1, 2]]$ s.t.

$$\mathbf{S}[f, [1, 2]] \subset \bigcup_{j \geq 1} (\tau_j - 5\epsilon(\tau_j)^\gamma, \tau + 5\epsilon(\tau_j)^\gamma)$$

$(\tau_j - \epsilon(\tau_j)^\gamma, \tau + \epsilon(\tau_j)^\gamma)$ pairwise disjoint

•Then

$$\tfrac{1}{2}\eta_1 \sum_{j\ge 1} \epsilon(\tau_j)^{3-\gamma} \le \sum_{j\ge 1} \int_{\tau_j-\epsilon(\tau_j)^\gamma}^{\tau_j} \dots$$

$$\le \int_0^2 \left(\int |\nabla_v (f(t,v)^{1/q} - 1)_+|^q dv \right)^{2/q} dt < \infty$$

•Since $\gamma = \frac{5q-6}{2q-2}$, one has $\frac{3-\gamma}{\gamma} = \frac{q}{5q-6}$, and the inequality above proves that

$$\mathcal{H}^{\frac{q}{5q-6}}(\mathbf{S}[f,[1,2]]) < \infty \quad \text{for each } q \in (\tfrac{4}{3}, 2)$$

Open question Is $\mathcal{H}^{\frac{1}{2}}(\mathbf{S}[f,[0,T]]) < \infty$? this is known to be true for Leray solutions of the Navier-Stokes equation in $\mathbf{R}^3$

Final Remarks/Perspectives

•The Desvillettes theorem puts the Landau equation in the same class as $3d$ Navier-Stokes in terms of Lebesgue exponents — except for the $(1+|v|)^{-3}$ weight

Navier-Stokes	$u \in L^\infty_t L^2_x\,,$	$\nabla_x u \in L^2_t L^2_x$
Landau	$\sqrt{f} \in L^\infty_t L^2_v\,,$	$\nabla_v \sqrt{f} \cap L^2_t L^2_{-3}$

•This suggests that a partial regularity theorem in (t, v) à la Caffarelli-Kohn-Nirenberg [CPAM 1982]+Vasseur [NoDEA 2007] might be within reach — in the case of the Navier-Stokes equation in $\mathbf{R}^3$, the 1-dimensional Hausdorff measure of the singular set of suitable solutions (Leray solutions with local energy inequality) is 0

•Hausdorff measures, Covering theorems

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•De Giorgi Method

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