Probabilistic limit theorems for chaotic dynamical systems, some results for dispersive billiards and Lorentz gases

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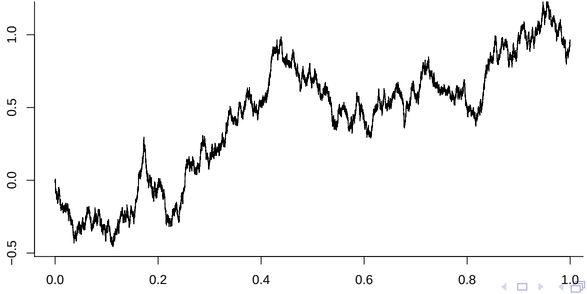
FCLT for sums of i.i.d. random variables and Wiener process

Let $(X_k)_k$ be a sequence of centered $\mathbb{R}$-valued i.i.d. random variables.

- **(SLLN)** $\frac{1}{n}\sum_{k=1}^n X_k \xrightarrow[n\to+\infty]{a.s.} 0$
- **(FCLT)** If $\mathbb{E}[X_1^2]<\infty$, then

$$(*) \quad \forall t_0>0, \quad \left(n^{-\frac{1}{2}}\sum_{k=1}^{\lfloor nt\rfloor} X_k\right)_{t\in[0,t_0]} \xrightarrow[n\to+\infty]{\mathcal{L},\ J_1} (cW_t)_{t\in[0,t_0]},$$

with $(W_t)_t$ a standard Wiener process.



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  with $(W_t)_t$ a standard Wiener process.
- If $\lim_{x\to\infty} x^2\mathbb{P}(\pm X_1\ge x)=A_\pm$, $A_+ + A_- > 0$, then

$$(*)\quad \forall t_0>0,\quad \left(\frac{1}{\sqrt{n\log n}}\sum_{k=1}^{\lfloor nt\rfloor} X_k\right)_{t\in[0,t_0]} \xrightarrow[n\to+\infty]{\mathcal{L},\ J_1} (cW_t)_{t\in[0,t_0]}\ .$$

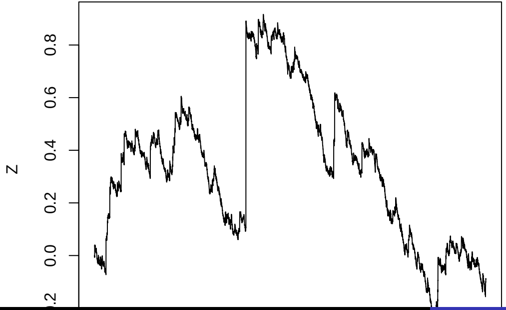
W is 2-self similar ($(W_{a^2t})_t$ has same distribution as $(aW_t)_t$), with independent stationary increments.

FCLT for sums of i.i.d. random variables and Lévy process

Let $\alpha \in (1,2)$. If X_k i.i.d. centered random variables such that $\lim_{x\to\infty} x^\alpha \mathbb{P}(\pm X_1 \geq x) = A_\pm$ with $A_+ + A_- > 0$, then for all $t_0 > 0$,

$$(*) \quad \left(n^{-\frac{1}{\alpha}} \sum_{k=1}^{\lfloor nt \rfloor} X_k \right)_{t\in[0,t_0]} \xrightarrow[n\to+\infty]{\mathcal{L},\ J_1} (\mathcal{Z}_t)_{t\in[0,t_0]} \ ,$$

$\mathcal{Z}$ is càdlàg (right continuous with left limits) and α-self similar ($(\mathcal{Z}_{a^\alpha t})_t$ has same distribution as $(a\mathcal{Z}_t)_t$) with independent stationary increments. A trajectory with $\alpha = 3/2$ and $A_- = 0$:



Probabilistic limit theorems for dynamical systems

We consider:

- a probability space $(\Omega, \mathcal{F}, \mathbb{P})$
- a transformation $T : \Omega \to \Omega$ preserving some probability measure μ. This means that $\mathbb{E}_\mu[f \circ T^k] = \mathbb{E}_\mu[f]$.
- we assume that $\mathbb{P} \ll \mu$
- a function $f : \Omega \to \mathbb{R}$ μ-centered

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- **(SLLN)** If the system is ergodic, then $\frac{S_n(f)}{n} \xrightarrow[n \to +\infty]{a.s.} 0$.
- [Zweimüller 01] The FCLT for $S_n(f)$ with respect to some probability measure $\mathbb{P}_0 \ll \mu$ implies the FCLT for $S_n(f)$ with respect to $\mathbb{P}$.

A simple example of chaotic dynamical system

$T : [0,1[\to [0,1[$ given by $T(x) = 10x \mod 1$.

T preserves the Lebesgue measure: $\int_0^1 f(T(x))\,dx = \int_0^1 f(x)\,dx$.

- This system is ergodic

$$\forall f \in L^1, \quad \sum_{k=0}^{n-1} f \circ T^k \xrightarrow[n\to+\infty]{a.s.} \int_0^1 f(x)\,dx\,.$$

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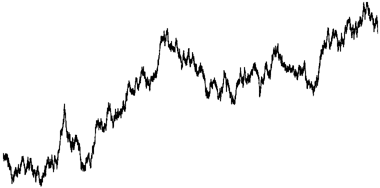
- If $f(x) = \lfloor 10x \rfloor$ (first digit), the r.v. $(X_k := f \circ T^k,\ k \geq 0)$ are i.i.d. with uniform distribution on $\{0, ..., 9\}$ and so

$$\text{FCLT :} \quad \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} (f \circ T^k - 4.5) \xrightarrow[n\to+\infty]{\mathcal{L},\ J_1} (\|f - 4.5\|_{L^2} W_t)_{t\in[0,t_0]}$$



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- More generally, if f is smooth enough and centered

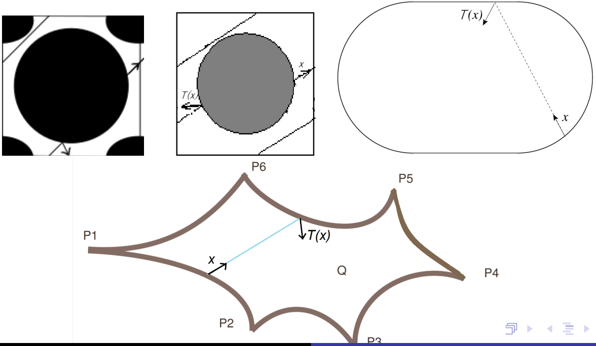
$$\left(\frac{1}{\sqrt{n}}\sum_{k=0}^{\lfloor nt\rfloor-1} f\circ T^k\right)_{t\in[0,t_0]} \xrightarrow[n\to+\infty]{\mathcal{L},\ J_1} \left(\sqrt{\sigma^2(f)}W_t\right)_{t\in[0,t_0]},$$

with $\sigma^2(f) := \sum_{n\in\mathbb{Z}} \int_0^1 f(x).f(T^{|n|}(x))\,dx$.

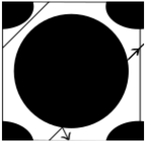
Examples: dispersing billiards

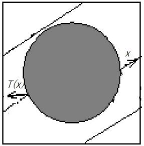
We consider a domain $Q\subset\mathbb{R}^2$ or $\mathbb{T}^2$ and a point particle moving in Q, going straight, with elastic reflections (reflected angle=incident angle).
We focus on the dynamics of such a point particle at reflection times.
Ω : set of unit reflected vectors, $T(x)$ next reflected vector,
T preserves μ with density $\rho:(q,\vec v)\mapsto\frac{1}{2|\partial Q|}\sin(\mathcal{T}_q\partial Q,\vec v)$.

Figure: Sinai Billiard in the torus with finite and with infinite horizon, Bunimovich stadium billiard & billiard with corners and cusps



Sinai Billiard in the torus with finite or infinite horizon





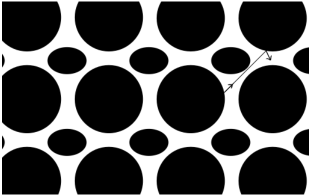
- [Sinai 1970] This system is ergodic.
- If f is Hölder continuous and μ-centered, then $\left(\frac{1}{\sqrt{n}}\sum_{k=0}^{\lfloor nt\rfloor -1} f\circ T^k\right)_{t\in[0,t_0]} \xrightarrow[n\to+\infty]{\mathcal{L},\ J_1} (\sigma(f)W_t)_{t\in[0,t_0]}$, with $\sigma^2(f)=\sum_{m\in\mathbb{Z}}\mathbb{E}_\mu\left[f.f\circ T^m\right]$.
[Bunimovich Sinai 1981, Bunimovich Chernov Sinai 1991, Young 1998, Chernov 1999]

$\mathbb{Z}^2$-periodic Lorentz gas

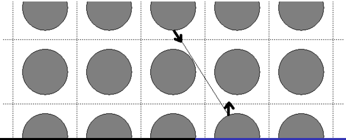
Consider a $\mathbb{Z}^2$-periodic configuration of say elliptic obstacles. Recall we consider a point particle:

- moving straight between obstacles
- with elastic reflection off obstacles (reflected angle=incident angle)

Finite horizon: It is not possible to go to infinity without hitting an obstacle.

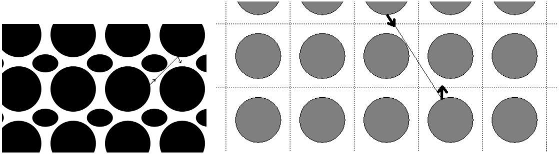


Infinite horizon: There are non parallel corridors of lines hitting no obstacle



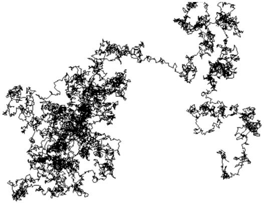
Functional Central limit theorem for the position

randomness: choose the initial position and speed randomly wrt $\mathbb{P} \ll Leb$.



The position q_n at the n-th reflection converges to a Brownian motion:

- Finite horizon: $\left(\frac{q_{\lfloor nt \rfloor}}{\sqrt{n}}\right)_t \xrightarrow[n\to+\infty]{\mathcal{L}} (\Sigma W_t)_t$ [BS1981,BCS1991,Y1998]



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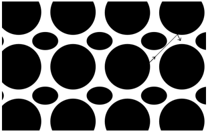
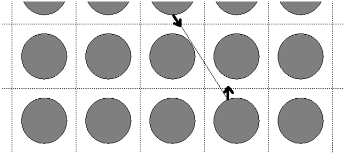

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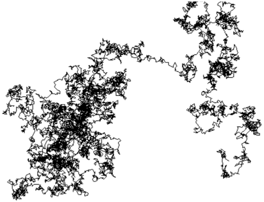
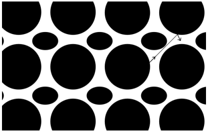
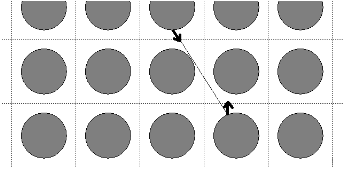
- Infinite horizon: $\left(\frac{q_{\lfloor nt \rfloor}}{\sqrt{n \log n}}\right)_t \xrightarrow[n\to+\infty]{\mathcal{L}} (\Sigma W_t)_t$ [SzászVarjú 2007]



Return time in a neighbourhood of the initial position

Let τ be the return time in the initial cell.
Let τ_ε be the number of reflections before coming back to an ε-neighbourhood of the initial position-speed.

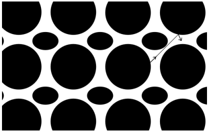
- Finite horizon[Dolgopyat Szász Varjú 08],[P. Saussol 10]

$$\mu(\tau > N) \sim_{N\to\infty} \frac{1}{c_0 \log N}, \quad \mu(\tau_\varepsilon > e^{\frac{t}{4\varepsilon^2\rho(\cdot)}}) \xrightarrow[\varepsilon\to 0]{} \frac{1}{1+c_0 t}.$$

- Infinite horizon[P. Terhesiu 21]

$$\mu(\tau > N) \sim_{N\to\infty} \frac{1}{c_0 \log\log N}.$$

$\mathbb{Z}^2$-Periodic pinball



Assume the particle wins a value $\beta_{\mathcal{O}}$ each time it hits the obstacle $\mathcal{O}$. Let Z_n be the sum won up to time n. Assume $\beta_{\mathcal{O}+\ell} = \beta_{\mathcal{O}}$ for all $\ell \in \mathbb{Z}^2$.

- $\frac{Z_n}{n} \overset{a.s.}{\underset{n\to+\infty}{\longrightarrow}} I_0(\beta) := \sum_{j=1}^{J} \beta_{\mathcal{O}_j}$.

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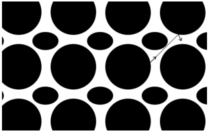

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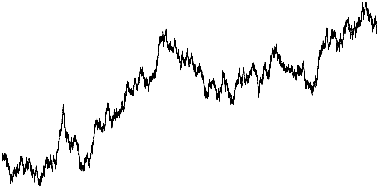
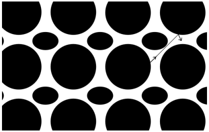
- $\frac{Z_n}{n} \xrightarrow[n\to+\infty]{a.s.} I_0(\beta) := \sum_{j=1}^{J} \beta_{\mathcal{O}_j}$.
- If $I_0(\beta) = 0$, then $\frac{Z_n}{\sqrt{n}} \xrightarrow[n\to+\infty]{\mathcal{L}} \sqrt{\sigma^2(\beta)} W_1$.

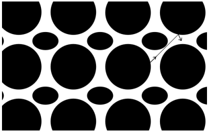




The particle wins a value $\beta_{\mathcal{O}}$ when it hits $\mathcal{O}$. Let Z_n be the sum won up to time n. Assume $\beta_{\mathcal{O}+(0,\ell)} = \beta_{\mathcal{O}}$ for all $\ell \in \mathbb{Z}$. Let $\eta > 0$.

- [Dolgopyat Szász Varjú 08] Assume that $\sum_{j=1}^{J} \sum_{\ell\in\mathbb{Z}} |\beta_{\mathcal{O}_j+(\ell,0)}| < \infty$.
  Then $\frac{Z_n}{\sqrt{n}} \xrightarrow[n\to+\infty]{\mathcal{L}} Y = \sum_{j=1}^{J} \sum_{\ell\in\mathbb{Z}} \beta_{\mathcal{O}_j+(\ell,0)} c_1 |W_1|$.

$\mathbb{Z}$-Periodic pinball



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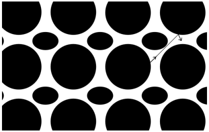
- [P. Thomine 19] If $\sum_{j=1}^{J} \sum_{\ell\in\mathbb{Z}} |\ell|^{\frac{1}{2}+\eta} |\beta_{\mathcal{O}_j+(\ell,0)}| < \infty$ and $Y = 0$, then $\frac{Z_n}{n^{\frac{1}{4}}} \xrightarrow[n\to+\infty]{\mathcal{L}} \sqrt{\widetilde{\sigma}^2(\beta) c_1 |W_1|}\, W_{-1}$.



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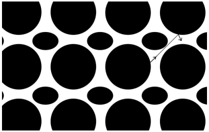
- follows from [Dombry,Guillotin-Plantard 09] (see [Phalempin]) If the $(\beta_{\mathcal{O}_1+(\ell,0)}, ..., \beta_{\mathcal{O}_J+(\ell,0)})$ are i.i.d. centered and L^2 (independent of the Lorentz gas), then $\frac{Z_n}{n^{\frac{3}{4}}} \xrightarrow[n\to+\infty]{\mathcal{L}} c_3$ Kesten-Spitzer process .

Pinball with non periodic gains



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Let Z_n be the sum won up to time n. Let $\eta > 0$.

- [Dolgopyat Szász Varjú 08] Assume $\sum_{\mathcal{O}} |\beta_{\mathcal{O}}| < \infty$, then

$$\frac{Z_n}{\log n} \xrightarrow[n\to+\infty]{\mathcal{L}} \sum_{\mathcal{O}} \beta_{\mathcal{O}} c_0 \mathcal{E} \text{ exponential r.v.}$$

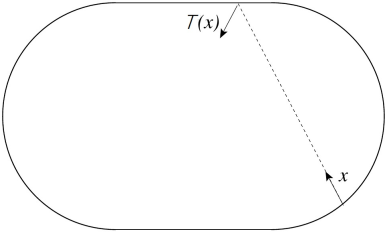
- [P. Thomine 19-21] If $\sum_{\mathcal{O}} d(0,\mathcal{O})^\eta |\beta_{\mathcal{O}}| < \infty$ and $\sum_{\mathcal{O}} \beta_{\mathcal{O}} = 0$, then

$$\frac{Z_n}{\sqrt{\log n}} \xrightarrow[n\to+\infty]{\mathcal{L}} \sqrt{\widetilde{\sigma}^2(\beta) c_0 \mathcal{E}}\, W_1 \,.$$

- [P. 09] If the $\beta_{\mathcal{O}}$ are i.i.d. centered and square integrable, then

$$\frac{Z_n}{\sqrt{n \log n}} \xrightarrow[n\to+\infty]{\mathcal{L}} c_4 W_1 .$$

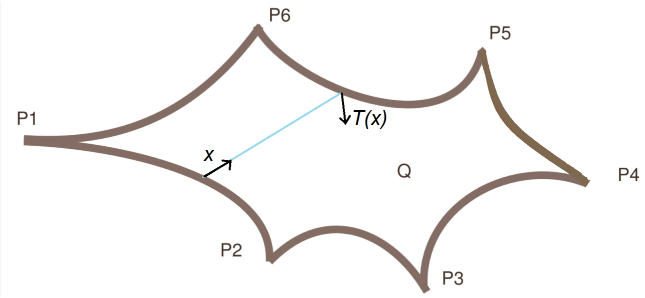
Bunimovich stadium billiard



[Bálint Gouëzel 06]
Let $f : \Omega \to \mathbb{R}$ be η-Hölder continuous and μ-centered. Then

$$\frac{\sum_{k=0}^{n-1} f \circ T^k}{\sqrt{n \log n}} \xrightarrow[n\to+\infty]{\mathcal{L}} \sigma_f W_1$$

with $\sigma_f = c_Q \int_I f(q,\uparrow)\,dq + \int_J f(q,\downarrow)\,dq$, I and J being the two flat segments (resp. at bottom and top) of the boundary of the stadium.



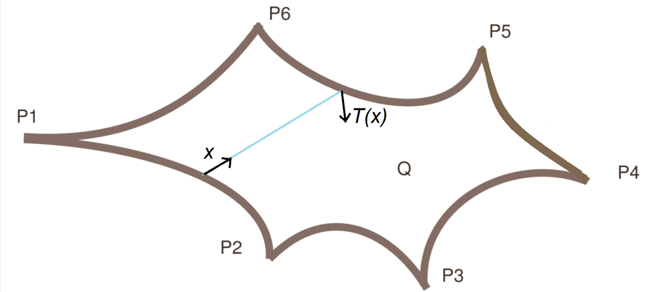
Let $f : \Omega \to \mathbb{R}$ be η-Hölder between two cusps/corners and μ-centered.

- 1 cusp given by two tangent circles. [Bálint Chernov Dolgopyat,11]

$$\left(\frac{\sum_{k=0}^{\lfloor nt\rfloor -1} f\circ T^k}{\sqrt{n\log n}}\right)_t \xrightarrow[n\to+\infty]{\mathcal{L},J_1} (\sigma_f W_t)_t,$$

$$\sigma_f := c.\int_{S^1} f(P,\vec{v})\,|\sin(\mathcal{T}_P Q,\vec{v})|^{\frac{1}{2}}\,d\vec{v}.$$

Dispersive billiards with cusps of higher order



Let $f : \Omega \to \mathbb{R}$ be η-Hölder between two cusps/corners and μ-centered.

- cusps: $z_{i,\pm}(s) = \pm c_{i,\pm} s^{\beta_i}/\beta_i + \mathcal{O}\left(s^{2\beta_i - 1}\right)$, $z'_{i,\pm}(s) = \pm c_{i,\pm} s^{\beta_i} + \mathcal{O}\left(s^{2\beta_i - 2}\right)$, with $c_{i,\pm} \geq 0$ not both 0, $\beta_* := \max \beta_i > 2$, $\alpha := \frac{\beta_*}{\beta_* - 1}$.
- Then $\left(n^{-\frac{1}{\alpha}} \sum_{k=0}^{\lfloor nt \rfloor - 1} f \circ T^k\right)_t \xrightarrow[n \to +\infty]{\mathcal{L},\ M_1} \left(\mathcal{Z}_t = \sum_{i\,:\,\beta_i = \beta_*} \sigma_{f,P_i} \mathcal{Z}_t^{(i)}\right)_t$ $\mathcal{Z}^{(i)}$ independent α-stable processes with independent and stationary increments. [Jung Zhang 18], [P. Young Zhang 20], [Melbourne Varandas 20]

Analytic tools behind several results mentioned in this talk

1. Consider P the dual of $g \mapsto g \circ T$: $\int_\Omega P(h).g\, d\mu = \int_\Omega h.g \circ T\, d\mu$.
2. Quasi-compactness of P: $P^n(h) = \int_\Omega h\, d\mu + \mathcal{O}(e^{-an})$ in $\mathcal{L}(\mathcal{B})$ with $\theta \in (0,1)$, for some nice complex Banach space $\mathcal{B}$, for some $a > 0$.
3. Set $P_t(h) = P(e^{it\Phi} h)$ and use characteristic functions:

$$\mathbb{E}_\mu[e^{it.\frac{S_n(\Phi)}{a_n}}] \approx \mathbb{E}_\mu[P^n_{t/a_n}(\mathbf{1})] .$$

4. Deduce from 2 by spectral perturbation method that $P_t^n = \lambda_t^n \Pi_t(\cdot) + \mathcal{O}(\theta_0^n)$ in $\mathcal{L}(\mathcal{B})$ with $\lim_{t\to 0} \lambda_t = 1$ in $\mathbb{C}$, with $\lim_{t\to 0} \Pi_t = \mathbb{E}_\mu[\cdot]$ in either $\mathcal{L}(\mathcal{B})$ or $\mathcal{L}(\mathcal{B}, L^1(\mu))$ and so

$$\mathbb{E}_\mu[e^{it.\frac{S_n(\Phi)}{a_n}}] \approx \lambda^n_{t/a_n} \mathbb{E}_\mu[\Pi_{t/a_n}(\mathbf{1})] \sim \lambda^n_{t/a_n}$$

5. E.g. if $\lambda_{t/a_n} \sim e^{-\frac{t^2}{2}}$, then $\frac{S_n(\Phi)}{a_n} \xrightarrow[n\to+\infty]{\mathcal{L}} W_1$ and

$$\mu(S_n(\Phi) = 0) = \frac{1}{2\pi} \int_{[-\pi,\pi[} \mathbb{E}_\mu[e^{it.S_n(\Phi)}]\, dt \approx \frac{1}{2\pi} \int_{[-\pi,\pi[^d} \lambda_t^n\, dt$$
$$\approx \frac{1}{2\pi a_n} \int_{\mathbb{R}} \lambda^n_{t/a_n}\, dt \approx \frac{1}{2\pi a_n} \int_{\mathbb{R}} e^{-\frac{t^2}{2}}\, dt = \frac{1}{\sqrt{2\pi}\, a_n}$$