# A mathematical introduction to the fluctuation theorem in non-equilibrium statistical mechanics

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From kinetic equations to statistical mechanics

2 July 2021

# Outline

# Second law of thermodynamics and time reversal

**Question:** *Is it possible to see the direction of time?*

**Clausius (1865):** *If a closed system is at some instant in a non-equilibrium macroscopic state, the* ***most probable*** *consequence at later instants is a steady increase in the* ***entropy*** *of the system.*

*Most probable* means that the probability of transition to states of higher entropy is so enormous in comparison with that of any appreciable decrease in entropy that in practice the latter can never be observed in Nature; see Landau–Lifshitz, V – § 8.

**Time reversal:** *Comparison of the statistical behaviour of trajectories under the forward and backward evolution*.

**New questions:**

*What is the* ***entropy****? How to characterise the* ***most probable****? What is the* ***statistical behaviour****?*

# Evolution of measures under Hamiltonian flows

Let us consider the Hamiltonian system

$$\dot{x} = \nabla_p H(x,p), \quad \dot{p} = -\nabla_x H(x,p), \quad (x,p) \in \mathbb{R}^{2d}, \qquad (1)$$

where $H(x,p)$ is a given smooth function. We assume that all the solutions of (1) are defined globally, so that we have a flow $\varphi_t : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ associated with (1).

## Definition

Given a measure $\mu$ on $\mathbb{R}^{2d}$, we define its image under $\varphi_t$ by

$$\mu_t(\Gamma) = \mu(\varphi_{-t}(\Gamma)) \quad \Longleftrightarrow \quad \langle f, \mu_t \rangle = \int_{\mathbb{R}^{2d}} f\big(\varphi_t(x,p)\big)\, \mu(\mathrm{d}x, \mathrm{d}p).$$

A measure $\mu$ is said to be invariant if $\mu_t = \mu$ for any $t \in \mathbb{R}$.

By Liouville's theorem, the Lebesgue measure is invariant.

# Image of absolutely continuous measures

Suppose that $\mu$ is absolutely continuous with respect to the Lebesgue measure:

$$\mu(\mathrm{d}x, \mathrm{d}p) = \rho(x, p)\, \mathrm{d}x\mathrm{d}p, \quad \rho \in C(\mathbb{R}^{2d}).$$

In this case, for any $t \in \mathbb{R}$ and $f \in C_0(\mathbb{R}^{2d})$, we can write

$$\begin{aligned} \langle f, \mu_t \rangle &= \int_{\mathbb{R}^{2d}} f\big(\varphi_t(x,p)\big)\rho(x,p)\, \mathrm{d}x\mathrm{d}p \\ &= \int_{\mathbb{R}^{2d}} f(y,q)\rho\big(\varphi_{-t}(y,q)\big) \underbrace{\mathrm{Jac}\big(\varphi_{-t}(y,q)\big)}_{=1} \mathrm{d}y\mathrm{d}q \\ &= \langle f, \rho \circ \varphi_{-t} \rangle. \end{aligned}$$

Thus, $\mu_t$ is also absolutely continuous:

$$\mu_t(\mathrm{d}x, \mathrm{d}p) = \rho\big(\varphi_{-t}(x,p)\big)\, \mathrm{d}x\mathrm{d}p.$$

# Gibbs measures

## Proposition

*If* $F : \mathbb{R}^{2d} \to \mathbb{R}$ *is an integral of motion for* (1),

$$F\big(\varphi_t(x,p)\big) = F(x,p) \quad \textit{for any} \quad t \in \mathbb{R}, (x,p) \in \mathbb{R}^{2d},$$

*then* $\mu(\mathrm{d}x, \mathrm{d}p) = F(x,p)\mathrm{d}x\mathrm{d}p$ *is preserved under the flow of* (1).

In particular, since $H$ is always an integral of motion, for any $\beta \geq 0$, the following measure is invariant:

$$\mu^{\beta}(\mathrm{d}x, \mathrm{d}p) := Z_{\beta}^{-1} e^{-\beta H(x,p)}\,\mathrm{d}x\mathrm{d}p.$$

It is called the Gibbs measure at the temperature $T = \beta^{-1}$.

# Coupled Hamiltonian systems

Consider two Hamiltonian systems coupled by a potential $V$:

$$\dot{x}_k = \nabla_{p_k} H_k(x_k, p_k), \quad \dot{p}_k = -\nabla_{x_k}\big(H_k(x_k, p_k) + V(x_1, x_2)\big), \quad (2)$$

where $(x_k, p_k) \in \mathbb{R}^{2d_k}$ and $k = 1, 2$. The Hamiltonian is given by

$$H(x, p) = H_1(x_1, p_1) + H_2(x_2, p_2) + V(x_1, x_2).$$

Consider the measure

$$\mu^\beta = \mu^{\beta_1} \otimes \mu^{\beta_2} = (Z_{\beta_1} Z_{\beta_2})^{-1} e^{-H^\beta(x,p)}\, \mathrm{d}x\mathrm{d}p, \quad \beta = (\beta_1, \beta_2),$$

where $H^\beta(x, p) = \beta_1 H_1(x_1, p_1) + \beta_2 H_1(x_2, p_2)$. This measure is invariant if and only if

$$\{H^\beta, H_1 + H_2 + V\} = 0 \quad \Longleftrightarrow \quad \{H^\beta, V\} = 0,$$

where $\{\cdot, \cdot\}$ denote the Poisson brackets.

# Logarithmic density and entropy

Suppose that $\mu^\beta$ is not invariant. Then, for any $t \in \mathbb{R}$,

$$\mu_t^\beta(\mathrm{d}x, \mathrm{d}p) = (Z_{\beta_1} Z_{\beta_2})^{-1} e^{-H^\beta(\varphi_{-t}(x,p))} \,\mathrm{d}x\mathrm{d}p.$$

Logarithmic density of the forward and backward images of $\mu^\beta$ can be written as:

$$\begin{aligned}\log \frac{\mathrm{d}\mu_t^\beta}{\mathrm{d}\mu_{-t}^\beta}(x,p) &= H^\beta(\varphi_t(x,p)) - H^\beta(\varphi_{-t}(x,p)) \\ &= \int_{-t}^{t} \frac{\mathrm{d}}{\mathrm{d}s} H^\beta\left(\varphi_s(x,p)\right) \mathrm{d}s \\ &= \sum_{k=1}^{2} \int_{-t}^{t} T_k^{-1} \underbrace{\{H_k, V\}}_{\text{energy flux}} \left(\varphi_s(x,p)\right) \mathrm{d}s.\end{aligned}$$

The right-most term is the total entropy produced on the time interval $[-t, t]$.

# Setting

Let $(\Omega_T, \mathcal{F}_T, \mathbb{P}_T)_{T\geq 1}$ be a sequence of probability spaces and let $\theta_T : \Omega_T \to \Omega_T$ be a measurable involution: $\theta_T \circ \theta_T = Id_{\Omega_T}$.

## Hypothesis (Regularity)

*The measures $\mathbb{P}_T$ and $\overline{\mathbb{P}}_T = \mathbb{P}_T \circ \theta_T^{-1}$ are equivalent $\forall T \geq 1$.*

Let us define the entropy production by

$$\sigma_T = \log \frac{d\mathbb{P}_T}{d\overline{\mathbb{P}}_T}.$$

Then $\sigma_T$ is a real-valued random variable on $\Omega_T$, and we denote by $P_T \in \mathcal{P}(\mathbb{R})$ its law under $\mathbb{P}_T$:

$$P_T(\Gamma) = \mathbb{P}_T\{\sigma_T \in \Gamma\}, \quad \Gamma \in \mathcal{B}(\mathbb{R}).$$

**Goal:** to investigate universal properties of these objects.

# Jarzynski's inequality and identity

**Jarzynski inequality:**

$$\int_{\mathbb{R}} s\, P_T(\mathrm{d}s) = \mathbb{E}_T \sigma_T \geq 0.$$

To see this, set $\varphi(x) = x \log x$ and use Jensen's inequality:

$$\int_{\mathbb{R}} s\, P_T(\mathrm{d}s) = \int_{\Omega_T} \left(\log \frac{\mathrm{d}\mathbb{P}_T}{\mathrm{d}\overline{\mathbb{P}}_T}\right) \frac{\mathrm{d}\mathbb{P}_T}{\mathrm{d}\overline{\mathbb{P}}_T} \mathrm{d}\overline{\mathbb{P}}_T \geq \varphi\left(\int_{\Omega_T} \frac{\mathrm{d}\mathbb{P}_T}{\mathrm{d}\overline{\mathbb{P}}_T} \mathrm{d}\overline{\mathbb{P}}_T\right) \geq 0.$$

**Jarzynski identity:**

$$\int_{\mathbb{R}} e^{-s}\, P_T(\mathrm{d}s) = 1.$$

$$\int_{\mathbb{R}} e^{-s}\, P_T(\mathrm{d}s) = \int_{\Omega_T} e^{-\sigma_T} \mathrm{d}\mathbb{P}_T = \int_{\Omega_T} \frac{\mathrm{d}\overline{\mathbb{P}}_T}{\mathrm{d}\mathbb{P}_T} \mathrm{d}\mathbb{P}_T = 1$$

# Transient fluctuation relation (TFR)

Define the Rényi entropy by

$$e_T(\alpha) = \mathrm{Ent}_\alpha(\mathbb{P}_T|\overline{\mathbb{P}}_T) := \log \int_{\Omega_T} \left(\frac{d\overline{\mathbb{P}}_T}{d\mathbb{P}_T}\right)^\alpha d\mathbb{P}_T = \log \int_{\Omega_T} e^{-\alpha\sigma_T} d\mathbb{P}_T.$$

**Evans–Searles fluctuation relation:**

$$e_T(1-\alpha) = e_T(\alpha) \quad \text{for any } \alpha \in \mathbb{R}.$$

It suffices to use a symmetry property for the Rényi entropy:

$$\begin{aligned} e_T(1-\alpha) &= \mathrm{Ent}_{1-\alpha}(\mathbb{P}_T|\overline{\mathbb{P}}_T) = \mathrm{Ent}_\alpha(\overline{\mathbb{P}}_T|\mathbb{P}_T) = \log \int_{\Omega_T} e^{\alpha\sigma_T} d\overline{\mathbb{P}}_T \\ &= \log \int_{\Omega_T} e^{\alpha\sigma_T\circ\theta_T} d(\overline{\mathbb{P}}_T \circ \theta_T) = \log \int_{\Omega_T} e^{-\alpha\sigma_T} d\mathbb{P}_T \\ &= e_T(\alpha). \end{aligned}$$

# Equivalent form of TFR

Let us denote by $\overline{P}_T$ the law of $\sigma_T \circ \theta_T = -\sigma_T$, so that $\overline{P}_T$ is the image of $P_T$ under the reflection $s \mapsto -s$; i.e. $\overline{P}_T(\Gamma) = P_T(-\Gamma)$.

**Equivalence of $P_T$ and $\overline{P}_T$:**

$$\frac{\mathrm{d}P_T}{\mathrm{d}\overline{P}_T}(s) = e^s \quad \text{for all } s \in \mathbb{R}.$$

Proof.

The TFR implies that, for any $\alpha \in [0,1]$,

$$\int_{\mathbb{R}} e^{-\alpha s} P_T(\mathrm{d}s) = \int_{\mathbb{R}} e^{-(1-\alpha)s} P_T(\mathrm{d}s) = \int_{\mathbb{R}} e^{-\alpha s} (e^s \overline{P}_T)(\mathrm{d}s).$$

The function $z \mapsto \int_{\mathbb{R}} e^{-zs} P_T(\mathrm{d}s)$ is analytic in the strip $0 < \mathrm{Re}\, z < 1$ and continuous on its closure. Since the characteristic function defines uniquely the corresponding measure, we conclude that $P_T(\mathrm{d}s) = (e^s \overline{P}_T)(\mathrm{d}s)$. □

# Large deviations and fluctuation relation

Let us consider the sequence $\{T^{-1}\sigma_T\}_{T\geq 1}$ under the law $\mathbb{P}_T$.

## Definition

We say that the LDP holds for the time-average of the entropy production if $\exists$ an l.s.c. function $I:\mathbb{R}\to[0,+\infty]$ (called rate function) such that

$$-I(\dot{\Gamma}) \leq \liminf_{T\to\infty} T^{-1}\log\mathbb{P}_T\{T^{-1}\sigma_T\in\Gamma\} \leq \limsup_{T\to\infty} T^{-1}\log\mathbb{P}_T\{T^{-1}\sigma_T\in\Gamma\} \leq -I(\overline{\Gamma}),$$

where $\overline{\Gamma}/\dot{\Gamma}$ is the closure/interior of $\Gamma\in\mathcal{B}(\mathbb{R})$, and $I(A)=\inf_A I$.

## Theorem (Gallavotti–Cohen fluctuation relation)

*Suppose the LDP holds for $\{T^{-1}\sigma_T\}_{T\geq 1}$ with a rate function $I$. Then $I(-s)=I(s)+s$ for any $s\in\mathbb{R}$.*

# Proof of the theorem

We know that $P_T(\mathrm{d}s) = e^s \overline{P}_T(\mathrm{d}s)$. It follows that

$$P_T(\Gamma) \le e^{\sup \Gamma}\, \overline{P}_T(\Gamma).$$

By the LDP and the relation $\overline{P}_T(\Gamma) = P_T(-\Gamma)$, we have

$$\begin{aligned} -I(\dot{\Gamma}) &\le \liminf_{T\to\infty} T^{-1} \log \mathbb{P}_T\{\sigma_T \in T\Gamma\} = \liminf_{T\to\infty} T^{-1} \log P_T(T\Gamma) \\ &\le \limsup_{T\to\infty} T^{-1} \log P_T(T\Gamma) \le \limsup_{T\to\infty} T^{-1} \log\{e^{T \sup \Gamma}\, \overline{P}_T(T\Gamma)\} \\ &\le \sup \Gamma - I(-\overline{\Gamma}) \quad \Longrightarrow \quad I(-\overline{\Gamma}) \le \sup \Gamma + I(\dot{\Gamma}). \end{aligned}$$

Taking $\Gamma = (r-\varepsilon, r+\varepsilon)$, we derive

$$\inf_{|s-r|\le\varepsilon} I(-s) \le r + \varepsilon + \inf_{|s-r|<\varepsilon} I(s),$$

whence, by l.s.c., $I(-r) \le r + I(r)\ \forall r$. This implies the result.

# Finite state Markov chains

Let $P = (p_{ij})_{i,j=1}^N$ be a stochastic matrix and let $\mu = (\mu_i)_{i=1}^N$ be a stationary distribution:

$$\sum_{j=1}^N p_{ij} = 1, \quad \sum_{j=1}^N \mu_j p_{ji} = \mu_i \quad \text{for } 1 \le i \le N.$$

We wish to study the difference between the forward and backward evolutions in the stationary regime. Let $\mathbb{P}_T$ be the path measure on $[0, T]$:

$$\mathbb{P}_T(\omega_0, \ldots, \omega_T) = \mu_{\omega_0} \prod_{t=1}^T p_{\omega_{t-1}\omega_t}.$$

The family $\{\mathbb{P}_T\}$ defines a unique measure $\mathbb{P}$ on the product space $\mathcal{A}^{\mathbb{Z}_+} = \Omega$.

# Recurrence and symmetry of transitions

## Hypothesis (Strong recurrence)

*There are $m \geq 1$ and $j_0 \in [1, N]$ such that*

$$p_{ij_0}^{(m)} > 0 \quad \text{for } 1 \leq i \leq N.$$

## Hypothesis (Symmetry of transitions)

*For any $i, j \in [1, N]$, we have $p_{ij} > 0$ if and only if $p_{ji} > 0$.*

*Strong recurrence* ensures the uniqueness of stationary distribution, while *Symmetry of transitions* implies the equivalence of laws for the forward and backward processes: defining the *time reversal* $\theta_T[\omega_0, \ldots, \omega_T] = [\omega_T, \ldots, \omega_0]$, we write

$$\overline{\mathbb{P}}_T = \mathbb{P}_T \circ \theta_T^{-1}, \quad \overline{\mathbb{P}}_T(\omega_0, \ldots, \omega_T) = \mu_{\omega_T} \prod_{t=1}^{T} p_{\omega_t \omega_{t-1}}.$$

# Entropy production

Define the density

$$\Delta_T(\omega_0, \ldots, \omega_T) = \frac{\mathrm{d}\mathbb{P}_T}{\mathrm{d}\overline{\mathbb{P}}_T}(\omega_0, \ldots, \omega_T).$$

Then the explicit formulas for the path measures implies that

$$\Delta_T(\omega_0, \ldots, \omega_T) = \frac{\mu_{\omega_0}}{\mu_{\omega_T}} \exp\left\{\sum_{t=1}^{T} \sigma(\omega_{t-1}, \omega_t)\right\},$$

$$\sigma(i,j) = \begin{cases} \log \frac{p_{ij}}{p_{ji}} & \text{if } p_{ij} > 0, \\ -\infty & \text{if } p_{ij} = 0. \end{cases}$$

The family $\{\overline{\mathbb{P}}_T\}_{T \geq 0}$ is consistent (in Kolmogorov's sense), so there is a unique probability measure $\overline{\mathbb{P}}$ on $\Omega = \mathcal{A}^{\mathbb{Z}_+}$ whose projections coincide with $\overline{\mathbb{P}}_T$ for any $T \geq 1$.

# Large-time behaviour

## Theorem

- *Dichotomy: Either $\mathbb{P} \perp \overline{\mathbb{P}}$ or $\mathbb{P} = \overline{\mathbb{P}}$. The latter holds iff the mean entropy production is zero:*

$$\bar{\sigma} = \int_{\Omega} \sigma \mathrm{d}\mathbb{P} = \sum_{i,j=1}^{N} \sigma(i,j)\mu_i p_{ij} = 0. \tag{3}$$

- *Detailed balance: $\mathbb{P} = \overline{\mathbb{P}}$ iff $\mu_i p_{ij} = \mu_j p_{ji}$ for all $1 \leq i,j \leq N$.*
- *Large deviations: Let $\bar{\sigma} > 0$, and $\exists m$ such that $p_{ij}^{(m)} > 0$ for all $(i,j)$. Then*

$$T^{-1}\sigma_T(\omega) = T^{-1}\sum_{t=1}^{T} \sigma(\omega_{t-1}, \omega_t), \;\; T \geq 1$$

*satisfies the LDP with convex good rate function $I$.*

- *Gallavotti–Cohen FR: $I(-r) = I(r) + r$ for all $r \in \mathbb{R}$.*

# Exponential separation in the non-reversible case

For $\gamma \in (0,1)$, define the Stein error exponent

$$\mathfrak{s}_T^{\gamma} = \min\{\overline{\mathbb{P}}_T(\Gamma_T) : \Gamma_T \subset \mathcal{A}^{T+1}, \mathbb{P}_T(\Gamma_T^c) \leq \gamma\},$$

and for $\theta > 0$, define the Hoeffding error exponent

$$\mathfrak{h}_\theta = \inf\Big\{\lim_{T\to\infty} T^{-1}\log \overline{\mathbb{P}}_T(\Gamma_T) : \limsup_{T\to\infty} T^{-1}\log \mathbb{P}_T(\Gamma_T^c) \leq -\theta\Big\}.$$

### Theorem

*Let us assume that $\bar{\sigma} > 0$. Then, for any $\gamma \in (0,1)$ and $\theta > 0$,*

$$\lim_{T\to+\infty} T^{-1}\log \mathfrak{s}_T^{\gamma} = -\bar{\sigma},$$

$$\mathfrak{h}_\theta = -\inf\{I(s) + s : s \in \mathbb{R}, I(s) < \theta\}.$$

# Setting

Let $\mathcal{A}$ be a finite alphabet, $\Omega = \mathcal{A}^{\mathbb{N}}$, and $\mathbb{P} \in \mathcal{P}(\Omega)$. Consider a mapping $\varphi : \Omega \to \Omega$ defined by $\varphi(\omega)_j = \omega_{j+1}$.

## Hypothesis (Upper and lower decoupling (ULD))

There is a sequence of positive numbers $\{C_T\}$ such that

$$\mathbb{P}\big(\omega_{[1,T+T']}\big) \geq C^{-1}_{T\wedge T'}\mathbb{P}\big(\omega_{[1,T]}\big)\mathbb{P}\big(\omega_{[T+1,T+T']}\big),$$
$$\mathbb{P}\big(\omega_{[1,T+T']}\big) \leq C_{T\wedge T'}\mathbb{P}\big(\omega_{[1,T]}\big)\mathbb{P}\big(\omega_{[T+1,T+T']}\big),$$
$$\lim_{T\to\infty} T^{-1}\log C_T = 0,$$

where $\omega_{[1,T]} = (\omega_1, \ldots, \omega_T)$ is either a $T$-tuple of points of $\mathcal{A}$ or the cylindrical set defined by it, and $T \wedge T' = \min\{T, T'\}$.

Let us define the empirical measures

$$\mu^{\omega}_T(\cdot) = \frac{1}{T}\sum_{t=0}^{T-1}\delta_{\varphi^t(\omega)}(\cdot) \in \mathcal{P}(\Omega), \quad T \geq 1, \quad \omega \in \Omega$$

# Level-3 large deviations and contraction

## Theorem

*Let $\mathbb{P} \in \mathcal{P}_\varphi(\Omega)$ be such that the upper and lower decoupling hypothesis holds. Then the empirical measures $\{\mu_T^{\cdot}\}_{T\geq 1}$ considered under the law $\mathbb{P}$ satisfy the LDP with a convex good rate function $\mathbb{I} : \mathcal{P}(\Omega) \to [0, +\infty]$; that is, $\forall\, \Gamma \in \mathcal{B}(\mathcal{P}(\Omega))$*

$$-\mathbb{I}(\dot{\Gamma}) \leq \liminf_{T\to\infty} \frac{1}{T} \log \mathbb{P}\{\mu_T^{\cdot} \in \Gamma\} \leq \liminf_{T\to\infty} \frac{1}{T} \log \mathbb{P}\{\mu_T^{\cdot} \in \Gamma\} \leq -\mathbb{I}(\overline{\Gamma}).$$

Let $V : \Omega \to \mathbb{R}$ be a continuous observable, where $\Omega$ is endowed with the Tikhonov topology. Then we can write

$$s_T(V)(\omega) := \frac{1}{T} \sum_{t=0}^{T-1} V(\varphi^t(\omega)) = \int_\Omega V(y)\mu_T^\omega(\mathrm{d}y) =: \langle V, \mu_T^\omega \rangle.$$

The function $\mu \mapsto \langle V, \mu \rangle$ is continuous from $\mathcal{P}(\Omega)$ to $\mathbb{R}$, and the contraction principle implies the LDP for $s_T(V)$.

# Level-3 Fluctuation relation

Define the space $\Omega_T = \mathcal{A}^T$ and the time reversal $\theta_T : \Omega_T \to \Omega_T$ by the relation $\theta_T(\omega_1, \dots, \omega_T) = (\omega_T, \dots, \omega_1)$ and suppose that the measures $\mathbb{P}_T$ and $\overline{\mathbb{P}}_T = \mathbb{P}_T \circ \theta_T^{-1}$ are equivalent for $T \geq 1$. Introduce the entropy production

$$\sigma_T(\omega) = \log \frac{\mathbb{P}_T(\omega_1, \dots, \omega_T)}{\overline{\mathbb{P}}_T(\omega_1, \dots, \omega_T)}.$$

## Theorem

*Under the above hypotheses, for any $\mathbb{Q} \in \mathcal{P}_\varphi(\Omega)$ there is a limit*

$$\mathrm{ep}(\mathbb{Q}) = \lim_{T\to\infty} T^{-1}\langle \sigma_T, \mathbb{Q}\rangle.$$

*Moreover, if $\mathbb{Q}_T$ and $\overline{\mathbb{Q}}_T$ are equivalent for any $T \geq 1$, $\mathbb{I}(\mathbb{Q}) < \infty$, and $\mathbb{I}(\overline{\mathbb{Q}}) < \infty$, then*

$$\mathbb{I}(\overline{\mathbb{Q}}) = \mathbb{I}(\mathbb{Q}) + \mathrm{ep}(\mathbb{Q}).$$

# LDP and FR for the entropy production

The mean entropy production $T^{-1}\sigma_T$ cannot be represented as an ergodic average of a continuous function.

## Theorem

*Let the ULD be satisfied for some* $\mathbb{P} \in \mathcal{P}_\varphi(\Omega)$ *and let the measures* $\mathbb{P}_T$ *and* $\overline{\mathbb{P}}_T$ *be equivalent. Then:*

**Pressure:** *For any* $\alpha \in \mathbb{R}$ *there is a finite limit*

$$e(\alpha) = \lim_{T\to\infty} T^{-1} \log\langle e^{-\alpha\sigma_T}, \mathbb{P}\rangle.$$

**LDP:** $\{T^{-1}\sigma_T\}$ *satisfies the LDP with the good rate function*

$$I(s) = \sup_{\alpha\in\mathbb{R}} \big(s\alpha - e(-\alpha)\big).$$

**Contraction:** *I satisfies the FR and can be written as*

$$I(s) = \inf\{\mathbb{I}(\mathbb{Q}) : \mathbb{Q} \in \mathcal{P}_\varphi(\Omega), \mathrm{ep}(\mathbb{Q}) = s\}$$

# 1D spin system

A spin system is defined by a potential

$$\boldsymbol{\Phi} = \{\Phi_X : \mathcal{A}^X \to \mathbb{R}, X \subset \mathbb{Z} \text{ is finite}\}.$$

We assume that $\boldsymbol{\Phi}$ is translation invariant and summable:

$$\Phi_X\big((\omega_j)_{j\in X}\big) = \Phi_{X-m}\big((\omega_{j+m})_{j\in X-m}\big), \quad \sum_{0\in X} \|\Phi_X\|_\infty < \infty.$$

The potential $\boldsymbol{\Phi}$ defines a family of conditional probabilities:

$$U_{\Lambda,\eta}(\omega) = \sum_{X\cap\Lambda\neq\varnothing} \Phi_X\big((\omega\vee\eta)_X\big),$$

$$Z_\Lambda(\eta) = \sum_{\omega\in\mathcal{A}^\Lambda} \exp\big(-U_{\Lambda,\eta}(\omega)\big),$$

$$P_\Lambda(\omega|\eta) = Z_\Lambda(\eta)^{-1} \exp\big(-U_{\Lambda,\eta}(\omega)\big).$$

# DLR equation and Gibbs measure

Let us set $\Omega = \mathcal{A}^{\mathbb{Z}}$, $\Omega_\Lambda = \mathcal{A}^\Lambda$, and given a measure $\mu \in \mathcal{P}(\Omega)$, denote by $\mu_\Lambda$ its projection to $\Omega_\Lambda$.

## Definition

We say that $\mu \in \mathcal{P}(\Omega)$ is a Gibbs measure if it satisfies the Dobrushin–Lanford–Ruelle equation for any finite $\Lambda \subset \mathbb{Z}$:

$$\mu_\Lambda(\omega) = \int_{\Omega_{\Lambda^c}} P_\Lambda(\omega|\eta)\mu_{\Lambda^c}(\mathrm{d}\eta) \quad \forall \omega \in \Omega_\Lambda.$$

Under the above hypotheses, there is a translationally invariant Gibbs measure. However, it is not necessarily unique.

## Proposition (Eizenberg–Kifer–Weiss (1994))

*Any Gibbs measure satisfies the ULD hypothesis.*

In particular, our results are true for the Gibbs measures.

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