Stochastic (Backward) Differential Equation with constraint in law

(aka Mean-Reflected (B)SDEs)

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joint works with {Briand, Guillin, Labart} & {Briand, Cardaliaguet, Hu}

A Stochastic Backward Excursion with Ying Hu

MR-SDE: financial introduction

Consider on $[0, T]$, $T > 0$ the system:

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}B_t \qquad X_0 = x_0,$$

- stochastic dynamics for the value of a portfolio X through the time X_t until a given date $T > 0$

MR-SDE: financial introduction

Consider on $[0, T]$, $T > 0$ for some $H : \mathcal{P}_2(\mathbf{R}^d) \to \mathbf{R}$, the system:

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t \qquad X_0 = x_0, \quad \mu_t = \mathrm{Law}(X_t)$$

$$\forall t \in [0, T] : \ H(\mu_t) \geqslant 0$$

- stochastic dynamics for the value of a portfolio X through the time X_t until a given date $T > 0$
- constrains to remain acceptable $\rightsquigarrow$ Apply the rule up to an event $\beta > 0$ (β small) *i.e.* $\mathbf{P}(X_t \geqslant m) \geqslant 1 - \beta \Leftrightarrow H(\mu_t) \geqslant 0$, $\quad H(\mu) = \langle \mathbf{1}_{x \geqslant m} - (1 - \beta), \mu \rangle$

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- $\forall r>0: \ K_{t+r}-K_t$ amount of cash added in the portfolio between time t and $t+r$ to keep the position acceptable

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- Introduced by Briand, Elie, Hu (Backward system), Considered in the same time by Jabir (forward) with different motivations/approach/results, Extension in the backward case by Djehiche, Elie, Hamadène.

The Skorohod problem (61)

For y a continuous path, find a continuous (x, k) s.t.

- (i) $\quad x_t = x_0 + y_t + k_t, \quad x_0 \geqslant 0$
- (ii) $\quad x_t \geqslant 0$
- (iii) $\quad t \mapsto K_t$ increasing and $K_0 = 0$
- (iv) "flatness": $\forall t > 0, \quad \int_0^t x_s dK_s = 0$

- Solution ?

  Set $H_t(k) := x_0 + y_t + k_t$.

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- Randomness ?

  Works for any continuous path, y may be a B.M.

The Skorohod problem revisited (Briand - Elie - Hu, 17)

Find a continuous (X, K), K deterministic s.t.

- (i) $X_t = x_0 + \int_0^t \beta - \alpha X_s ds + \sigma B_t + K_t, \quad x_0 \geqslant p$
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- Solution ?
  - Set $Y_t = x_0 + \int_0^t \beta - \alpha Y_s ds + \sigma B_t \rightsquigarrow \mathbf{E}[Y_t] = e^{-\alpha t} x_0 + \frac{\beta}{\alpha}(1 - e^{-\alpha t})$

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  - Solution is $K_t = \int_0^t e^{-\alpha s} d\nu_s, \quad \nu_s := \sup_{r \leqslant s} \max(0, -e^{\alpha r}(\mathbf{E}[Y_r] - p))$

MR-SDE: Mathematical set-up, another point of view

Consider on $[0, T]$, $T > 0$ and for some $H : \mathcal{P}_2(\mathbf{R}^d) \to \mathbf{R}$ the system:

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- Smooth coefficients, let φ be a test function,

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\frac{\mathrm{d}}{\mathrm{d}t}E[\varphi(X_t)] &= \mathbf{E}\left[(b(X_t)\partial_y + \frac{1}{2}\sigma^2(X_t)\partial_y^2)\varphi(X_t)\right] + \mathbf{E}\left[\partial_y\varphi(X_t)\right]\mathrm{d}K_t \\
&= \int_{\mathbf{R}}(b(y)\partial_y + \frac{1}{2}\sigma^2(y)\partial_y^2)\varphi(y)\mathrm{d}\mu_t(y) + \int_{\mathbf{R}}\partial_y\varphi(y)\mathrm{d}\mu_t(y)\mathrm{d}K_t
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- Integrate by parts: we obtain (distributional sense)

$$\partial_t\mu_t - \frac{1}{2}\partial^2(\sigma^2\mu_t) - div(\mu_t b) - div(\mu_t)\mathrm{d}K_t = 0, \quad \mu_0 = \delta_{x_0}$$

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with K **deterministic**

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↳ MR-SDE ⇝ Skorokhod problem on Fokker-Planck PDE !

MR-SDE scalar case + scalar constraint

Consider on $[0, T]$, $T > 0$ for some $h : \mathbf{R} \to \mathbf{R}$ the system:

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where B is a Brownian motion define on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P})$ with K deterministic

- How the process K looks like?

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- How the process K looks like?
  - Let Y be the solution on $[0, T]$ of $\mathrm{d}Y_t = b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}B_t, \quad Y_0 = x_0$

$$\hookrightarrow X_t = Y_t + K_t \implies \mathbf{E}[h(X_t)] = \mathbf{E}[h(Y_t + K_t)]$$

  - Set $H_t(K_t) := \mathbf{E}[h(Y_t + K_t)]$

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- •K_t ?

MR-SDE scalar case + scalar constraint

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- h bi-Lipschitz $\rightsquigarrow G_0^+ : \mathcal{P}(\mathbf{R}) \ni \mu \mapsto G_0^+(\mu)$ Lipschitz for the Wasserstein distance !

MR-SDE scalar case: Mean field counterpart ($\mu_0 = \delta_{x_0}$)

$$\begin{cases} X_t = x_0 + \int_0^t b(X_s)\mathrm{d}s + \int_0^t \sigma(X_s)\mathrm{d}B_s + K_t \\ E[h(X_t)] \geqslant 0, \quad \int_0^t \mathbf{E}[h(X_s)]\mathrm{d}K_s = 0 \\ \end{cases}$$

- Mean reflection ↔ non linear reflection (McKean-Vlasov sense)
  ↳ Interacting reflected particles system with oblique reflection ?

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Constrained in law SDE : a more general setting

Consider on $[0, T]$, $T > 0$ the system:

$$X_t = X_0 + \int_0^t b(X_s)\mathrm{d}s + \int_0^t \sigma(X_s)\mathrm{d}B_s + \int_0^t dK_s$$
$$\forall t \in [0, T] : \ \mathbf{E}h(X_t) \geqslant 0 \quad \int_0^t \mathbf{E}[h(X_s)]\mathrm{d}K_s = 0$$

At the PDE level: (in distribution sense)

$$\partial_t \mu_t - \frac{1}{2}\partial^2(\sigma^2 \mu_t) + div(\mu_t b) + div(\mu_t)\mathrm{d}K_t = 0, \quad \mu_0 = \delta_x$$

$$+\forall t \in [0, T], \quad H(\mu_t) := \int h \mathrm{d}\mu_t \geqslant 0, \quad \int_0^t H(\mu_s)\mathrm{d}K_s = 0$$

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$$\partial_t \mu_t - \frac{1}{2}\partial^2(\sigma^2 \mu_t) + div(\mu_t b) + div(\mu_t)\mathrm{d}K_t = 0, \quad \mu_0 = \delta_x$$

$$+\forall t \in [0, T], \quad H(\mu_t) := \int h \mathrm{d}\mu_t \geqslant 0, \quad \int_0^t H(\mu_s)\mathrm{d}K_s = 0$$

- Extension to the multi-dimensional setting ?

Constrained in law SDE : a more general setting

Consider on $[0,T]$, $T>0$ the system:

$$X_t = X_0 + \int_0^t b(X_s)\mathrm{d}s + \int_0^t \sigma(X_s)\mathrm{d}B_s + \int_0^t dK_s$$
$$\forall t \in [0,T]: \; H(\mu_t) \geqslant 0 \quad \int_0^t H(\mu_s)\mathrm{d}K_s = 0$$

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- Extension to the multi-dimensional setting ?
- More general reflections : $H : \mathcal{P}_2(\mathbf{R}^d) \to \mathbf{R}^d$?

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↳ no representation formula for K anymore!

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↳ no representation formula for K anymore!

↳ consider "normal" reflection

Constrained in law SDE with normal reflection

Consider on $[0, T]$, $T > 0$ the system:

$$
\begin{aligned}
&X_t = X_0 + \int_0^t b(X_s)\mathrm{d}s + \int_0^t \sigma(X_s)\mathrm{d}B_s + \partial_\mu H(\mu_s)(X_s)\mathrm{d}K_t \\
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\end{aligned}
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$$\forall t \in [0, T]: \ H(\mu_s) \geqslant 0 \quad \int_0^t H(\mu_s)\mathrm{d}K_s = 0$$

- Coefficients Lispchitz + H is "bi-Lipschitz" : there exist $0 < m \leqslant M$ such that

$$\forall \mu \in \mathcal{P}^2(\mathbf{R}^d) \qquad m^2 \leqslant \int_{\mathbf{R}^d} |\partial_\mu H(\mu)|^2(x)\, \mu(dx) \leqslant M^2,$$

  + H has "Lispchitz" derivative : there exists $C \geqslant 0$ such that

$$\forall X \in L^2(\Omega), \qquad \mathbf{E}\left[|\partial_\mu H([X])(X) - \partial_\mu H([Y])(Y)|^2\right] \leqslant C\, \mathbf{E}\left[|X - Y|^2\right],$$

  + H partially C^2

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  + H partially C^2

AND

- the coefficient σ is uniformly bounded
- OR H is L-concave: $H(\nu) - H(\mu) - \mathbf{E}\left[\partial_\mu H([X])(X)\cdot(Y-X)\right] \leqslant 0.$
  (Carmona-Delarue / Jabir) "curve is below the tangent"

Result: The normally reflected SDE admits a unique solution (X,K)

Constrained in law SDE with normal reflection: penalization

Consider on $[0, T]$, $T > 0$ the system:

$$
\begin{aligned}
&X_t = X_0 + \int_0^t b(X_s)\mathrm{d}s + \int_0^t \sigma(X_s)\mathrm{d}B_s + \partial_\mu H(\mu_s)(X_s)\mathrm{d}K_t \\
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Define the penalized approximation:

$$X_t^k = X_0 + \int_0^t b(X_s^k)\, ds + \int_0^t \sigma(X_s^k)\, dB_s + \int_0^t \partial_\mu H(\mu_s)(X_s^k)\, \psi_k(s, H(\mu_s))\, ds, \quad t \geqslant 0,$$

where

$\psi_k(t,x) = r(t)$ if $x \leqslant -1/k$, $\quad \psi_k(t,x) = -kr(t)x$, if $-1/k \leqslant x \leqslant 0$, $\quad \psi_k(t,x) = 0$, if x

Constrained in law SDE with normal reflection: penalization

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- $(X^k, K^k)_{k \geqslant 0}$ is a Cauchy sequence

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- $(X^k, K^k)_{k \geqslant 0}$ is a Cauchy sequence -> main step

Find $r(\cdot)$ so that $\forall k, \; \forall s \in [0, T]: \quad H(\mu_s^k) \geqslant -1/k$

↳ relies on "bi-Lipschitz" property + partial C^2 regularity of H
+ control of the second order term in Itô - measure expansion
(Chassagneux-Crisan-Delarue) ⇝ concavity of H OR boundedness of σ

Constrained in law SDE with normal reflection: PDE (2^{nd})

Consider on $[0, T]$, $T > 0$ the system:

$$X_t = X_0 + \int_0^t b(X_s)\mathrm{d}s + \int_0^t \sigma(X_s)\mathrm{d}B_s + K_t$$

$$\forall t \in [0, T]: \; H(\mu_t) \geqslant 0 \quad \int_0^t H(\mu_s)\mathrm{d}K_s = 0$$

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\end{aligned}
$$

- Let $\phi : \mathcal{P}_2(\mathbf{R})$ (smooth) and $u(t, \mu) := \phi(\mu_T)$

Constrained in law SDE with normal reflection: PDE (2^{nd})

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$$\begin{aligned} X_t &= X_0 + \int_0^t b(X_s)\mathrm{d}s + \int_0^t \sigma(X_s)\mathrm{d}B_s + K_t \\ \forall t\in[0,T]:\ \ H(\mu_t) &\geqslant 0 \quad \int_0^t H(\mu_s)\mathrm{d}K_s = 0 \end{aligned}$$

- Let $\phi:\mathcal{P}_2(\mathbf{R})$ (smooth) and $u(t,\mu) := \phi(\mu_T)$
- Itô's formula on $\mathcal{P}_2(\mathbf{R})$

$$\begin{cases} (\partial_t + \mathcal{A})u(t,\mu) = 0 \\ \\ u(T,\mu) = \phi(\mu) \end{cases}$$

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where

$$\mathcal{A}\varphi = \int b(z)\partial_\mu\varphi(\mu)(z)\mathrm{d}\mu(z) + \frac{1}{2}\int \sigma^2(z)\partial_z\partial_\mu\varphi(\mu)(z)\mathrm{d}\mu(z)$$

- Non linear process

Constrained in law SDE with normal reflection: PDE (2^{nd})

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$$
\begin{aligned}
X_t &= X_0 + \int_0^t b(X_s)\mathrm{d}s + \int_0^t \sigma(X_s)\mathrm{d}B_s + K_t \\
\forall t \in [0, T] &: \; H(\mu_t) \geqslant 0 \quad \int_0^t H(\mu_s)\mathrm{d}K_s = 0
\end{aligned}
$$

- Let $\phi : \mathcal{P}_2(\mathbf{R})$ (smooth) and $u(t, \mu) := \phi(\mu_T)$
- Itô's formula on $\mathcal{P}_2(\mathbf{R})$

$$
\begin{cases}
(\partial_t + \mathcal{A})u(t, \mu) = 0 \\
\text{Neumann} \\
u(T, \mu) = \phi(\mu)
\end{cases}
$$

where

$$
\mathcal{A}\varphi = \int b(z)\partial_\mu\varphi(\mu)(z)\mathrm{d}\mu(z) + \frac{1}{2}\int \sigma^2(z)\partial_z\partial_\mu\varphi(\mu)(z)\mathrm{d}\mu(z)
$$

- Non linear process

constrained $\rightarrow$ Neumann condition

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$$\forall t \in [0, T] : \ H(\mu_t) \geqslant 0 \quad \int_0^t H(\mu_s)\mathrm{d}K_s = 0$$

- Let $\phi : \mathcal{P}_2(\mathbf{R})$ (smooth) and $u(t, \mu) := \phi(\mu_T)$
- u is **A** viscosity solution of

$$\begin{cases} (\partial_t + \mathcal{A})u(t, \mu) = 0 \text{ if } H(\mu) > 0, \\ + \int \partial_\mu H(\mu)(z)\partial_\mu u(t, \mu)(z)\mathrm{d}\mu(z) = 0 \text{ if } H(\mu) = 0, \\ u(T, \mu) = \phi(\mu) \end{cases}$$

where

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- Non linear process
- non-linear constrained → Neumann condition on the L-derivative

Constrained in law SDE with normal reflection: Mean Field counterpart

Consider on $[0,T]$, $T>0$ the system:

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X_t^i &= X_0^i + \int_0^t b(s, X_s^i)\, ds + \int_0^t \sigma(s, X_s^i)\, dB_s^i + \int_0^t \partial_\mu H\left(\mu_s^N\right)(X_s^i)\, dK_s^N, \\
X_0^i &\sim \mu_0, \quad 1 \leqslant i \leqslant N \\
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$$\forall t \in [0, T] : \ H(\mu_s) \geqslant 0 \quad \int_0^t H(\mu_s)\mathrm{d}K_s = 0$$

- "natural" MF counterpart: Interacting particles system reflected in mean field

$$X_t^i = X_0^i + \int_0^t b(s, X_s^i)\, ds + \int_0^t \sigma(s, X_s^i)\, dB_s^i + \int_0^t \partial_\mu H\left(\mu_s^N\right)(X_s^i)\, dK_s^N,$$

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- MF Skorokhod problem ? $\rightsquigarrow$ NO : $H(\mu_0) \geqslant 0 \not\Longrightarrow H(\mu_0^N) \geqslant 0$

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  ↳ Transport initial conditions along the normal up to set of constraint

$$\rightsquigarrow \tilde{X}_0^i(r) = X_0^i + \int_0^r \partial_\mu H(\mu^N_{X_0^i(v)}(v))(\tilde{X}_0^i(v))dv$$

Constrained in law SDE with normal reflection: Mean Field counterpart

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- MF system converges: $\mathbf{E}X_0^4 < +\infty$ + (H concave OR σ bounded) $\rightsquigarrow$ $W_2(\mu^N_{X_0^i(r)}(r), \mu_0) \to 0$ and coupling works.

Constrained in law SDE : generalization

- Back to PDE pb: Skorokhod problem on deterministic fokker-planck

$$\partial_t \mu_t = \frac{1}{2}\partial^2([\sigma\sigma^*]\mu_t) + \mathrm{div}(\mu_t b) + \mathrm{div}(\partial_\mu H(\mu_t)\mu_t)\mathrm{d}K_t$$

$$H(\mu_t) \geqslant 0 \quad \int_0^t H(\mu_s)\, dK_s = 0, \quad t \geqslant 0$$

Constrained in law SDE : generalization

- Back to PDE pb: Skorokhod problem on stochastic fokker-planck

$$\partial_t \mu_t = \frac{1}{2}\partial^2([\sigma\sigma^* + \sigma_1\sigma_1^*]\mu_t) + \mathrm{div}(\mu_t b) + \mathrm{div}(\partial_\mu H(\mu_t)\mu_t)\mathrm{d}K_t$$
$$+\mathrm{div}\,(\mu_t \sigma_1 dW_t)$$
$$H(\mu_t) \geqslant 0 \quad \int_0^t H(\mu_s)\, dK_s = 0, \quad t \geqslant 0$$

- K is no longer deterministic.

Constrained in conditional law SDE : generalization

- Back to PDE pb: Skorokhod problem on stochastic fokker-planck

$$\partial_t \mu_t = \frac{1}{2}\partial^2([\sigma\sigma^* + \sigma_1\sigma_1^*]\mu_t) + \mathrm{div}(\mu_t b) + \mathrm{div}(\partial_\mu H(\mu_t)\mu_t)\mathrm{d}K_t$$
$$+\mathrm{div}\left(\mu_t \sigma_1 dW_t\right)$$
$$H(\mu_t) \geqslant 0 \quad \int_0^t H(\mu_s)\, dK_s = 0, \quad t \geqslant 0$$

- K is no longer deterministic. Associated SDE:

$$X_t = X_0 + \int_0^t b(s, X_s)\, ds + \int_0^t \sigma(s, X_s)\, dB_s + \int_0^t \sigma_1(s, X_s) dW_s$$
$$+ \int_0^t \partial_\mu H([X_s|W])(X_s)\, dK_s,$$
$$H([X_t|W]) \geqslant 0, \quad \int_0^t H([X_s|W])\, dK_s = 0, \quad t \geqslant 0.$$

Constrained in conditional law SDE with common noise

- Back to PDE pb: Skorokhod problem on stochastic fokker-planck

$$\partial_t \mu_t = \frac{1}{2}\partial^2([\sigma\sigma^* + \sigma_1\sigma_1^*]\mu_t) + \mathrm{div}(\mu_t b) + \mathrm{div}(\partial_\mu H(\mu_t)\mu_t)\mathrm{d}K_t$$
$$+\mathrm{div}\,(\mu_t\sigma_1 dW_t)$$
$$H(\mu_t) \geqslant 0 \quad \int_0^t H(\mu_s)\, dK_s = 0, \quad t \geqslant 0$$

- K is no longer deterministic. Associated SDE:

$$X_t = X_0 + \int_0^t b(s, X_s)\, ds + \int_0^t \sigma(s, X_s)\, dB_s + \int_0^t \sigma_1(s, X_s) dW_s$$
$$+ \int_0^t \partial_\mu H([X_s|W])(X_s)\, dK_s,$$
$$H([X_t|W]) \geqslant 0, \quad \int_0^t H([X_s|W])\, dK_s = 0, \quad t \geqslant 0.$$

- Associated MF system:

$$X_t^i = X_0^i + \int_0^t b(s, X_s^i)\, ds + \int_0^t \sigma(s, X_s^i)\, dB_s^i + \int_0^t \sigma_1(s, X_s^i)\, dW_s + \int_0^t \partial_\mu H\left(\mu_s^N\right)(X_s^i)\, dK_s^N,$$
$$\mu_t^N = \frac{1}{N}\sum_{i=1}^N \delta_{X_t^i}, \quad H\left(\mu_t^N\right) \geqslant 0, \quad \int_0^t H\left(\mu_s^N\right)\, dK_s^N = 0, \quad t \geqslant 0.$$

Constrained in law SDE: resume

- measurable in time and Lipschitz in space coefficients (could be Wasserstein w.r.t. a law argument)

- constraints of the form $H(\mu) \geqslant 0$ + H "bi-Lipschitz" (locally)
  ↳ one dimensional case and constraint of scalar type ($\exists!$) and MF counterpart ($\exists!$ +convergence $\rightsquigarrow$ numerical scheme)

- + H partially C^2 + H is concave OR σ is bounded
  ↳ normal reflection, any dimension ($\exists!$) and MF counterpart ($\exists!$ +convergence $\rightsquigarrow$ numerical scheme) + Neumann PDE on Wasserstein space ($\exists$ in viscosity sense + Feynman-Kak formulae)

- + H is fully C^2
  ↳ reflection in conditional law ($\exists!$) and MF counterpart with common noise + Stochastic Neumann PDE (Feynman-Kak formulae)

What about constrained in law backward SDE ?

- "same assumptions give same results" (up to common noise)
  ↳ Warning: works for H concave + fully C^2 (σ bounded $\rightsquigarrow$ Z bounded)
  ↳ Warning: convergence of MF counterpart requires Z bounded in $\mathbb{L}_p$, $p > 4$ $\rightsquigarrow$ requires regularity
  ↳ Warning: Associated PDE is not Neumann: driver independent of Z $\rightsquigarrow$ $\exists$ viscosity solution to obstacle problem on Wasserstein space + Feynman-Kak formulae for decoupling field

Thank you!