Quadratic Mean-Field Reflected BSDEs

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Plan

- BSDEs with constraints
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  - Quadratic case with unbounded terminal

BSDEs with constraints

Backward stochastic differential equations

A nonlinear BSDE is defined on a filtered probability space satisfying the usual conditions [Pardoux-Peng, Systems & Control Letters, 1990]:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \ t \in [0, T].$$

- The data:
  - ξ: the terminal condition, an $\mathcal{F}_T$-measurable r.v. such that $\mathbb{E}[|\xi|^2] < \infty$;
  - $f : [0, T] \times \Omega \times \mathbb{R}^2 \to \mathbb{R}$, the unif. Lip. generator, such that
    $$\mathbb{E}\left[\left(\int_0^T |f(s, 0, 0)|ds\right)^2\right] < \infty.$$
- There exists a unique adapted solution (Y, Z) such that
  $$\underbrace{\mathbb{E}\left[\sup_{t\in[0,T]} |Y_t|^2\right] < \infty}_{\mathcal{S}^2}, \quad \underbrace{\mathbb{E}\left[\int_0^T |Z_s|^2 ds\right] < \infty}_{\mathcal{H}^2}.$$
- Later, relax the condition, e.g., quadratic BSDEs [Kobylanski, AP. 00], [Briand-Hu, PTRF. 06, 08].

BSDEs with pathwise constraints

- 1-dimensional Reflected BSDE [El Karoui et al., AP, 97]
  - Reflecting boundary condition and "minimal" solution:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s + K_T - K_t, \ \ t \in [0, T],$$

with $Y_t \geqslant L_t$, where K is a non-decreasing adapted process satisfying the following **Skorohod condition**

$$\int_0^T (Y_t - L_t) dK_t = 0.$$

- General reflecting condition: $(Y_t, Z_t) \in \Gamma_t$, $dt \times \mathbb{P}(d\omega)$-a.s., e.g. [Cvitanić et al., AP, 98]; [Peng, PTRF, 00]; [Buckdahn-Hu, MOR, 98]; [Buckdahn-Hu, Adv in Appl. Prob., 98]; [Peng-Xu, Bernoulli, 10].
- Applications: e.g. constrained superhedging problems

BSDEs with mean constraints

- BSDE with mean reflection [Briand, Elie and Hu, AAP., 17]
  - The formulation of the BSDE with mean reflection is as follows:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s + K_T - K_t, \ t \in [0, T], \quad (1)$$

    with

$$\mathbb{E}[\ell(t, Y_t)] \geqslant 0, \qquad 0 \leqslant t \leqslant T,$$

    where ℓ is non-decreasing in y.
  - Deterministic flat solution: A triplet (Y, Z, K) satisfies (1) where K is non-decreasing and deterministic satisfying

$$\int_0^T \mathbb{E}[\ell(t, Y_t)]dK_t = 0.$$

- Applications: partial hedging of financial derivatives, e.g. $\ell = 1_{x \geqslant u_t} - v_t$, i.e. Y is required to beat deterministic continuous benchmark u_t with a probability greater than v_t at time t.

BSDEs with mean reflection

- Pathwise comparison fails (counterexample [Hibon-Hu-L-Luo-Wang, MCRF, 18]):

$$Y_t = \xi - \int_t^T Z_s dW_s + K_T - K_t, \ \mathbb{E}[Y_t] \geqslant 2T - t, \ t \in [0, T],$$

Consider the equation with two terminal conditions

$$\xi^1 = |W_T|^2 \text{ and } \xi^2 = \frac{3}{2}|W_T|^2.$$

Then, the solutions of two equations can be defined by

- $Y_t^1 = |W_t|^2 + 2T - 2t,$
  $Z_t^1 = 2W_t, \ K_t^1 = t;$
- $Y_t^2 = \frac{3}{2}|W_t|^2 + \max\left(2T - \frac{5}{2}t, \frac{3}{2}T - \frac{3}{2}t\right),$
  $Z_t^2 = 3W_t, \ K_t^2 = \min\left(t, \frac{T}{2}\right).$

It is obvious that pathwise comparison fails on $(0, \frac{T}{2}]$

$$Y_t^1 - Y_t^2 = \frac{t - |W_t|^2}{2}.$$

BSDEs with mean reflection

- BSDE with linear mean reflection:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s + K_T - K_t, \ t \in [0, T]$$

with

$$\mathbb{E}[Y_t] \geqslant \mu, \qquad 0 \leqslant t \leqslant T,$$

- Mean filed BSDEs:

$$Y_t^n = \xi + \int_t^T f(s, Y_s^n, Z_s^n)ds + n\int_t^T (\mu - \mathbb{E}[Y_s^n])_+ ds - \int_t^T Z_s^n dB_s].$$

- $(Y^n, Z^n, \int_0 (\mu - \mathbb{E}[Y_s^n])_+ ds)$ converges to (Y, Z, K).
- No piecewise comparison theorem, see Buckdahn, Li and Peng [SPA. 2009].

BSDEs with mean reflection

- Simple case:

$$Y_t = \xi + \int_t^T f_s ds - \int_t^T Z_s dW_s + K_T - K_t, \text{ with } \mathbb{E}[\ell(t, Y_t)] \geqslant 0.$$

- $Y_t = \overline{Y}_t + (K_T - K_t)$ with $\overline{Y}_t = \mathbb{E}_t[\xi + \int_t^T f_s ds]$.
- $\mathbb{E}[Y_t] = \mathbb{E}[\overline{Y}_t] + (K_T - K_t)$ with $\mathbb{E}[\ell(t, Y_t)] \geqslant 0$.
- Solution: $K_T - K_t = \sup_{t \leqslant s \leqslant T} \inf_{x \in \mathbb{R}_+} \{x : \mathbb{E}[\ell(s, x + \overline{Y}_s)] \geqslant 0\}$.
- Auxiliary function:

$$L_t : L^2(\mathcal{F}_T) \longrightarrow [0, \infty) \quad X \longmapsto \inf\{x \geqslant 0 : \mathbb{E}[\ell(t, x + X)] \geqslant 0\}.$$

- Assumption (L): The loss function $\ell : \Omega \times [0, T] \times \mathbb{R} \longrightarrow \mathbb{R}$ is measurable w.r.t. $\mathcal{F}_T \times \mathcal{B}([0, T]) \times \mathcal{B}(\mathbb{R})$, and there exists $C \geqslant 0$ such that
  - $(t, y) \longmapsto \ell(t, y)$ is continuous;
  - $\forall t \in [0, T]$, $y \longmapsto \ell(t, y)$ is strictly increasing;
  - $\forall t \in [0, T]$, $\mathbb{E}[\ell(t, \infty)] > 0$;
  - $\forall t \in [0, T]$, $\forall y \in \mathbb{R}$, $|\ell(t, y)| \leqslant C_\ell(1 + |y|)$.

BSDEs with mean reflection

- Solution map: $\Gamma : U \to \Gamma(U)$ with $\Gamma(U) = Y^U$:

$$Y_t^U = \xi + \int_t^T f(s, U_s, Z_s^U) ds - \int_t^T Z_s^U dW_s + K_T^U - K_t^U, \text{ with}$$

$$\mathbb{E}[\ell(t, Y_t^U)] \geqslant 0, \quad \int_0^T \mathbb{E}[\ell(t, Y_t^U)] dK_t^U = 0.$$

- Consider

$$y_t^U = \xi + \int_t^T f(s, U_s, z_s^U) ds - \int_t^T z_s^U dW_s;$$

- Representation result:

$$Y_t^U = y_t^U + \sup_{t \leqslant s \leqslant T} L_s(y_s^U), \ Z_t^U = z_t^u, \ K_t^U = \sup_{0 \leqslant s \leqslant T} L_s(y_s^U) - \sup_{t \leqslant s \leqslant T} L_s(y_s^U).$$

Some related research

- Lipschitz: Briand, Elie and Hu (AAP, 18), fixed point
- Quadratic with bounded terminal: Hibon-Hu-Lin-Luo-Wang (MCRF, 18), BMO martingale+fixed point;
- Quadratic with unbounded terminal: Hu-Moreau-Wang (JTP, 24), θ-method+convex generator;
- Multi-dimensional: Briand-Cardaliaguet-Chaudru de Raynal-Hu (SPA, 20):

$$\begin{cases} Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s + \int_t^T D_\mu H(\mathrm{P}_{Y_s})(Y_s)dK_s, \\ \mathrm{P}_{Y_t} \in \{\mu \in \mathcal{P}_2(\mathbb{R}^m), \ H(\mu) \geqslant 0\}, \ \int_0^T H(\mathrm{P}_{Y_t})dK_t = 0, \end{cases}$$

in which $D_\mu H$ denotes the Lions' derivative; Qu and Wang (EJP, 23): $\mathbb{E}[Y_t] \in D$ for a possible non-convex domain D.

- Mean-field reflected BSDE [Djehiche, Elie and Hamadène, AAP. 21]

  - The formulation of mean-field reflected BSDE is as follows:

$$\begin{cases} Y_t = \xi + \int_t^T f(s, Y_s, \mathrm{P}_{Y_s}, Z_s)ds - \int_t^T Z_s dB_s + K_T - K_t, \\ Y_t \geqslant h(t, Y_t, \mathrm{P}_{Y_t}), \quad \int_0^T (Y_t - h(t, Y_t, \mathrm{P}_{Y_t}))dK_t = 0, \end{cases} \tag{2}$$

    where P_{Y_t} is the marginal probability distribution of the process Y at time t.

  - Applications: motivated by applications in pricing life insurance contracts with surrender options
  - Example: $h = \gamma_1 Y_t + \gamma_2 \mathbb{E}[|Y_t|]$, then $(1-\gamma_1)\mathbb{E}[Y_t] \geqslant \gamma_2 \mathbb{E}[|Y_t|]$. Thus we need to assume

$$\gamma_1 + \gamma_2 \leqslant 1$$

    in general. Otherwise, it is unreasonable.
  - Variation constraints: $(Y_t - \mathbb{E}[Y_t])^2 \leqslant a^2 \to -a \leqslant Y_t - \mathbb{E}[Y_t] \leqslant a$.

Mean-field reflected BSDE

- Simple case (f is independent of z):

$$Y_t = \xi + \int_t^T f(s, Y_s, \mathrm{P}_{Y_s})ds - \int_t^T Z_s dW_s + K_T - K_t, \text{ with}$$

$$Y_t \geqslant h(t, Y_t, \mathrm{P}_{Y_t}), \quad \int_0^T (Y_t - h(t, Y_t, \mathrm{P}_{Y_t}))dK_t = 0.$$

- Solution map: $\Gamma : U \to \Gamma(U)$ with $\Gamma(U) = Y^U$:

$$Y_t^U = \xi + \int_t^T f(s, U_s, \mathrm{P}_{U_s})ds - \int_t^T Z_s^U dW_s + K_T^U - K_t^U, \text{ with}$$

$$Y_t^U \geqslant h(t, U_t, \mathrm{P}_{U_t}), \quad \int_0^T (Y_t^U - h(t, U_t, \mathrm{P}_{U_t}))dK_t^U = 0.$$

- Snell envelope representation:

$$\Gamma(U)_t = \operatorname*{ess\,sup}_{\tau \text{ stopping time} \geqslant t} \mathbb{E}_t\Big[\xi 1_{\{\tau = T\}} + h\left(\tau, U_\tau, (\mathrm{P}_{U_s})_{s=\tau}\right) 1_{\{\tau < T\}} + \int_t^\tau f(s, U_s, \mathrm{P}_{U_s})ds\Big]$$

Mean-field reflected BSDE

- Assumption (Lip):

$$|f(t,y_1,\nu_1,z_1)-f(t,y_2,\nu_2,z_2)| \leqslant \lambda\left(|y_1-y_2|+W_1(\nu_1,\nu_2)+|z_1-z_2|\right).$$

$$|h(t,y_1,\nu_1)-h(t,y_2,\nu_2)| \leqslant \gamma_1|y_1-y_2|+\gamma_2 W_1(\nu_1,\nu_2).$$

- The fixed point method:

$$\|\Gamma(U^1)-\Gamma(U^2)\|_{\mathcal{S}^p}$$

$$\leqslant (2T\lambda+\gamma_1+\gamma_2)^{\frac{p-1}{p}}\left(\left(\frac{p}{p-1}\right)^p(T\lambda+\gamma_1)+T\lambda+\gamma_2\right)^{\frac{1}{p}}\|U^1-U^2\|_{\mathcal{S}^p}.$$

- Enhanced sufficient condition:

$$(\gamma_1+\gamma_2)^{\frac{p-1}{p}}\left(\left(\frac{p}{p-1}\right)^p\gamma_1+\gamma_2\right)^{\frac{1}{p}}<1 \text{ (stronger than } \gamma_1+\gamma_2\leqslant 1)$$

Theorem (f is independent of z, Djehiche et al., AAP., 21)

Assume the above conditions hold and $\xi \in L^p$ for $p>1$. Then mean-filed reflected BSDEs admits a unique solution $(Y,Z,K)\in\mathcal{S}^p\times\mathcal{H}^p\times\mathcal{A}^p$.

Mean-field reflected BSDE

- f depends on z: we can define similarly the solution map $\Gamma(U) = Y^U$

$$Y_t^U = \xi + \int_t^T f(s, U_s, \mathrm{P}_{U_s}, Z_s^U)ds - \int_t^T Z_s^U dW_s + K_T^U - K_t^U, \text{ with}$$

$$Y_t^U \geqslant h(t, U_t, \mathrm{P}_{U_t}), \quad \int_0^T (Y_t^U - h(t, U_t, \mathrm{P}_{U_t}))dK_t^U = 0.$$

- Snell envelope representation:

$$\Gamma(U)_t = \operatorname*{ess\,sup}_{\tau \text{ stopping time} \geqslant t} \mathbb{E}_t\Big[\xi 1_{\{\tau=T\}} + h\left(\tau, U_\tau, (\mathrm{P}_{U_s})_{s=\tau}\right) 1_{\{\tau<T\}} + \int_t^\tau f(s, U_s, \mathrm{P}_{U_s}, Z_s^U)ds\Big]$$

- It is difficult to prove Γ is contraction due to Z, like $\|\cdots\| \leqslant C \times \|\cdots\|^{\frac{1}{2}}$.
- Thus [Djehiche et al., AAP., 21] applied a penalization method to obtain the existence under some additional assumptions, e.g. $f(s, Y_s, \mathrm{E}[Y_s], Z_s)$, $h((t, Y_t, \mathrm{E}[Y_t])$ are non-decreasing with respect to $\mathrm{E}[Y_t]$, since the comparison principle for mean-filed BSDE is quite restricted, see [Buckdahn, Li and Peng, SPA. 19].

Mean-field reflected BSDE

- Nonlinear Snell envelope representation
  - Given a reflected BSDE:

$$\begin{cases} Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Y_s dB_s + K_T - K_t, & 0 \leqslant t \leqslant T, \\ Y_t \geqslant L_t, \quad \forall t \in [0, T] \text{ and } \int_0^T (Y_t - L_t) dK_t = 0, \end{cases}$$

  - Consider a standard BSDE

$$y_t^\tau = \xi 1_{\{\tau = T\}} + L_\tau 1_{\{\tau < T\}} + \int_t^\tau f(s, y_s^\tau, z_s^\tau)ds - \int_t^\tau z_s^\tau dB_s,$$

  - Under appropriate condition, we have

$$Y_t = \operatorname*{ess\,sup}_{\tau \text{ stopping time} \geqslant t} \mathcal{E}_{t,\tau}^f [\xi 1_{\{\tau = T\}} + L_\tau 1_{\{\tau < T\}}], \quad \forall t \leqslant T,$$

    where (conditional g-expectation) $\mathcal{E}_{t,\tau}^f [\xi 1_{\{\tau = T\}} + L_\tau 1_{\{\tau < T\}}] := y_t^\tau$.
- The aim is to find a fixed point of the following map Γ:

$$\Gamma(U)_t := \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_t} \mathcal{E}_{t,\tau}^{f^U} [\xi 1_{\{\tau = T\}} + h(\tau, U_\tau, (\mathrm{P}_{U_s})_{s=\tau}) 1_{\{\tau < T\}}]$$

where the driver f^U is given by $f^U(t, z) := f(t, U_t, \mathrm{P}_{U_t}, z)$.

Lipschitz case

- (Djehiche, Dumitrescu and Zeng, Arxiv, 2021) (they consider mean-filed BSDEs with jump) used nonlinear Snell envelope representation to remove the additional conditions in [Djehiche et al., AAP., 21] under the following enhanced sufficient condition

$$\gamma_1^p + \gamma_2^p \leqslant 2^{2-\frac{3p}{2}}, \quad p \geqslant 2$$

which does not cover each other. They used Itô's formula to estimate the difference.

- There is also an additional domination condition [Assumption 2.1 (ii)(b) in Djehiche, Dumitrescu and Zeng, Arxiv, 2021] , that requires:

$$\text{The process } \sup_{(y,v)\in\mathbb{R}\times\mathcal{P}_1(\mathbb{R})} |h(t,y,v)| \text{ belongs to } \mathcal{S}^p,$$

which fails for general Lipshcitz h. Example: $h = \gamma_1|y|$ and $\sup_y |h(t,y)| = \infty$.

The idea: Girsanov's theorem

- Consider

$$y_t^{i,\tau} = \xi 1_{\{\tau=T\}} + h(\tau, U_\tau^i, (\mathrm{P}_{U_s^i})_{s=\tau}) 1_{\{\tau<T\}} + \int_t^\tau f(s, U_s^i, \mathrm{P}_{U_s^i}, z_s^{i,\tau}) ds - \int_t^\tau z_s^{i,\tau} dB_s.$$

- denote by

$$\beta_t = \frac{f(t, U_s^1, \mathrm{P}_{U_s^1}, z_t^{1,\tau}) - f(t, U_s^1, \mathrm{P}_{U_s^1}, z_t^{2,\tau})}{|z_t^{1,\tau} - z_t^{2,\tau}|^2} (z_t^{1,\tau} - z_t^{2,\tau}) 1_{\{|z_t^{1,\tau} - z_t^{2,\tau}| \neq 0\}}.$$

- $\widetilde{B}_t := B_t - \int_0^t \beta_s^\top 1_{[0,\tau]}(s) ds$, defines a Brownian motion under the equivalent probability measure $\widetilde{\mathrm{P}}$ given by $d\widetilde{\mathrm{P}} := \mathscr{E}(\beta 1_{[0,\tau]} \cdot B)_0^T d\mathrm{P}$:

$$\begin{aligned} y_t^{1,\tau} - y_t^{2,\tau} &= h(\tau, U_\tau^1, (\mathrm{P}_{U_s^1})_{s=\tau}) 1_{\{\tau<T\}} - h(\tau, U_\tau^2, (\mathrm{P}_{U_s^2})_{s=\tau}) 1_{\{\tau<T\}} \\ &- \int_t^T 1_{[0,\tau]}(s)(z_s^{1,\tau} - z_s^{2,\tau}) d\widetilde{B}_s \\ &\left(= \int_t^T (z_s^{1,\tau} - z_s^{2,\tau}) dB_s - \int_t^T f(s, U_s^1, \mathrm{P}_{U_s^1}, z_s^{1,\tau}) - f(s, U_s^1, \mathrm{P}_{U_s^1}, z_s^{2,\tau}) ds \right) \\ &+ \int_t^T f(s, U_s^1, \mathrm{P}_{U_s^1}, z_s^{2,\tau}) - f(s, U_s^2, \mathrm{P}_{U_s^2}, z_s^{2,\tau}) 1_{[0,\tau]}(s) ds. \end{aligned}$$

- We have

$$
\begin{aligned}
&\|\Gamma(U^1)-\Gamma(U^2)\|_{\mathcal{S}^p}\\
&\leqslant \exp\left(\frac{\lambda^2\sqrt{T}}{2}\right)(\gamma_1+\gamma_2+2\lambda T)^{\frac{p-1}{p}}\left((\gamma_1+\lambda T)\left(\frac{p}{p-1-\sqrt{T}}\right)^{\frac{p}{\sqrt{T}+1}}\right.\\
&\qquad\left.+(\gamma_2+\lambda T)\right)^{\frac{1}{p}}\times\|U^1-U^2\|_{\mathcal{S}^p}.
\end{aligned}
$$

Theorem (Hu, Moreau and Wang, PUQR, 22)

Assume the Lipschitz conditions and $\xi \in L^p$ for $p > 1$. If γ_1 and γ_2 satisfy

$$(\gamma_1+\gamma_2)^{\frac{p-1}{p}}\left(\left(\frac{p}{p-1}\right)^p\gamma_1+\gamma_2\right)^{\frac{1}{p}}<1,$$

then mean-field reflected BSDEs admits a unique solution.

Quadratic case with bounded terminal

(H1) The terminal condition $\xi \in \mathcal{L}^\infty$ with $\xi \geqslant h(T, \xi, \mathrm{P}_\xi)$.

(H2) There exist three positive constants α, β and γ such that for any $t \in [0, T]$, $y \in \mathbb{R}$, $v \in \mathcal{P}_1(\mathbb{R})$ and $z \in \mathbb{R}^d$

$$|f(t, y, v, z)| \leqslant \alpha + \beta(|y| + W_1(v, \delta_0)) + \frac{\gamma}{2}|z|^2.$$

(H3) The process $h(\cdot, y, v) \in \mathcal{S}^\infty$ is uniformly bounded with respect to (t, ω, y, v).

(H4) There exist two constants $\gamma_1, \gamma_2 > 0$ such that for any $t \in [0, T]$, $y_1, y_2 \in \mathbb{R}$, $v_1, v_2 \in \mathcal{P}_1(\mathbb{R})$

$$|h(t, y_1, v_1) - h(t, y_2, v_2)| \leqslant \gamma_1 |y_1 - y_2| + \gamma_2 W_1(v_1, v_2).$$

(H5) There exists a constant κ such that for each $t \in [0, T]$, $y_1, y_2 \in \mathbb{R}$, $v_1, v_2 \in \mathcal{P}_1(\mathbb{R})$ and $z_1, z_2 \in \mathbb{R}^d$

$$|f(t, y_1, v_1, z_1) - f(t, y_2, v_2, z_2)| \leqslant \beta\left(|y_1 - y_2| + W_1(v_1, v_2)\right) + \kappa(1 + |z_1| + |z_2|)|z_1 - z_2|.$$

Quadratic case with bounded terminal

- $\beta_t = \frac{f(t,U_s^1,\mathrm{P}_{U_s^1},z_t^{1,\tau}) - f(t,U_s^1,\mathrm{P}_{U_s^1},z_t^{2,\tau})}{|z_t^{1,\tau} - z_t^{2,\tau}|^2}(z_t^{1,\tau} - z_t^{2,\tau})1_{\{|z_t^{1,\tau} - z_t^{2,\tau}| \neq 0\}}$ is bounded in the BMO space and $d\widetilde{\mathrm{P}} := \mathscr{E}(\beta 1_{[0,\tau]} \cdot B)_0^T d\mathrm{P}$ a true martingale.
- We have

$$\|\Gamma(U^1) - \Gamma(U^2)\|_{\mathcal{S}^\infty} \leqslant (\gamma_1 + \gamma_2 + 2\lambda T)\|U^1 - U^2\|_{\mathcal{S}^\infty}.$$

Theorem (Hu, Moreau and Wang, PUQR, 22)

Assume (H1)-(H5). If γ_1 and γ_2 satisfy

$$\gamma_1 + \gamma_2 < 1,$$

then mean-field reflected BSDEs admits a unique solution $(Y, Z, K) \in \mathcal{S}^\infty \times BMO \times \mathcal{A}$.

Quadratic case with unbounded terminal

- $\mathbb{L}$ is the collection of all $\mathscr{F}_T$-measurable r.v. ξ satisfying $\mathbb{E}\left[e^{p|\xi|}\right] < \infty$ for any $p > 1$.
- $\mathbb{S}$ is the collection of all stochastic processes Y such that $e^Y \in \mathcal{S}^p$ for any $p > 1$.

(H1') The terminal condition $\xi \in \mathbb{L}$ with $\xi \geqslant h(T, \xi, \mathrm{P}_\xi)$.

(H3') For any $y \in \mathbb{R}$, $v \in \mathcal{P}_1(\mathbb{R})$, the process $h(t, y, v)$ belongs to $\mathbb{S}$.

(H5') For each $t \in [0, T]$, $y_1, y_2 \in \mathbb{R}$, $v_1, v_2 \in \mathcal{P}_1(\mathbb{R})$ and $z \in \mathbb{R}^d$

$$|f(t, y_1, v_1, z) - f(t, y_2, v_2, z)| \leqslant \beta\left(|y_1 - y_2| + W_1(v_1, v_2)\right).$$

(H6) $f(t, y, v, \cdot)$ is concave or convex.

Theorem (Hu, Moreau and Wang, PUQR, 22)

Assume (H1'), (H2), (H3'), (H4), (H5') and (H6) hold. If γ_1 and γ_2 satisfy $4(\gamma_1 + \gamma_2) < 1$, then mean-field reflected BSDEs admits a unique solution $(Y, Z, K) \in \mathbb{S} \times \mathcal{H}^d \times \mathcal{A}$.

The idea: θ-method [Briand and Hu, PTRF. 2008]

- for each $\theta \in (0,1)$ estimate $\Gamma(U^1) - \theta\Gamma(U^2)$ and $\Gamma(U^2) - \theta\Gamma(U^1)$ instead of $\Gamma(U^1) - \Gamma(U^2)$
  - Consider BSDE
$$Y_t^i = \xi + \int_t^T f(Z_s^i)ds - \int_t^T Z_s^i dB_s.$$
  - set $(\delta_\theta Y, \delta_\theta Z) := \left(\frac{Y^1-\theta Y^2}{1-\theta}, \frac{Z^1-\theta Z^2}{1-\theta}\right)$, we have
$$\delta_\theta Y_t = \delta_\theta Y_T + \int_t^T \delta_\theta f(\delta_\theta Z_s)ds - \int_t^T \delta_\theta Z_s dB_s,$$
    with $\delta_\theta f(z) = \frac{1}{1-\theta}\left(f((1-\theta)z + \theta Z_t^2) - \theta f(Z_t^2)\right)$.
  - $\delta_\theta f(z) \leqslant f(t,z) \leqslant \alpha + \frac{\gamma}{2}|z|^2$, the constants does not depend on θ and Z^2, and $e^{\gamma(\delta_\theta Y_t)^+ + \gamma\int_0^t \alpha_s ds}$ is a sub-martingale, i.e.,
$$e^{\gamma(\delta_\theta Y_t)^+} \leqslant \mathbb{E}_t[e^{\gamma\xi^+ + \gamma\int_t^T \alpha_s ds}].$$
  - For uniqueness, (note $(Y_t^1 - \theta Y_t^2)^+ = (1-\theta)(\delta_\theta Y_t)^+$)
$$\begin{aligned}(Y_t^1 - Y_t^2)^+ &\leqslant (Y_t^1 - \theta Y_t^2)^+ + (1-\theta)(Y_t^2)^- \\ &\leqslant (1-\theta)\left(\frac{1}{\gamma}e^{\gamma(\delta_\theta Y_t)^+} + (Y_t^2)^-\right).\end{aligned}$$

θ-method: Existence

- (Fan, Hu and Tang, JDE, 23) first used θ-method to study multi-D. quadratic BSDE with unbounded terminal.
- define recursively a sequence of stochastic processes $(Y^{(m)})_{m=1}^{\infty}$ through the following quadratic reflected BSDE (Bayraktar and Yao, SPA. 2012):

$$\begin{cases} Y_t^{(m)} = \xi + \int_t^T f(s, Y_s^{(m)}, \mathrm{P}_{Y_s^{(m-1)}}, Z_s^{(m)})ds - \int_t^T Z_s^{(m)} dB_s + K_T^{(m)} - K_t^{(m)}, \\ Y_t^{(m)} \geqslant h(t, Y_t^{(m-1)}, \mathrm{P}_{Y_t^{(m-1)}}), \int_0^T (Y_t^{(m)} - h(t, Y_t^{(m-1)}, \mathrm{P}_{Y_t^{(m-1)}}))dK_t^{(m)} = 0 \end{cases}$$

- combine nonlinear Snell envelope representation and θ-method:

$$Y_t^{(m)} := \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_t} y_t^{(m),\tau}$$

in which $y_t^{(m),\tau}$ is the solution of the following quadratic BSDE

$$\begin{aligned} y_t^{(m),\tau} =& \xi 1_{\{\tau=T\}} + h(\tau, Y_\tau^{(m-1)}, (\mathrm{P}_{Y_s^{(m-1)}})_{s=\tau}) 1_{\{\tau<T\}} \\ &+ \int_t^\tau f(s, y_s^{(m),\tau}, \mathrm{P}_{Y_s^{(m-1)}}, z_s^{(m),\tau})ds - \int_t^\tau z_s^{(m),\tau} dB_s. \end{aligned}$$

The sketch

- for any $p > 1$

$$\sup_{m\geqslant 0}\mathbb{E}\left[\exp\left\{p\gamma\sup_{s\in[0,T]}|Y_s^{(m)}|\right\}+\left(\int_0^T|Z_t^{(m)}|^2dt\right)^p+|K_T^{(m)}|^p\right]<\infty.$$

- for any $p \geqslant 1$, we have

$$\Pi(p):=\sup_{\theta\in(0,1)}\lim_{m\to\infty}\sup_{q\geqslant 1}\mathbb{E}\left[\exp\left\{p\gamma\sup_{s\in[0,T]}\delta_\theta\overline{Y}_s^{(m,q)}\right\}\right]<\infty,$$

where $\delta_\theta Y^{(m,q)}=\frac{\theta Y^{(m+q)}-Y^m}{1-\theta}$, $\delta_\theta\widetilde{Y}^{(m,q)}=\frac{\theta Y^{(m)}-Y^{(m+q)}}{1-\theta}$ and $\delta_\theta\overline{Y}:=|\delta_\theta Y^{(m,q)}|+|\delta_\theta\widetilde{Y}^{(m,q)}|$.

- for any $p \geqslant 1$ and $\theta\in(0,1)$,

$$\limsup_{m\to\infty}\sup_{q\geqslant 1}\|Y^{(m+q)}-Y^{(m)}\|_{\mathcal{S}^p}^p\leqslant 2^{p-1}(1-\theta)^p\left(\frac{\Pi(1)p!}{\gamma^p}+\sup_{m\geqslant 1}\|Y_t^{(m)}\|_{\mathcal{S}^p}^p\right).$$

Thank you for your attention!