Projections of semimartingales and BSDEs

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SI TU DIVISES TON ÂGE PAR 4
ET QUE TU MULTIPLIES LE DOUBLE DU RÉSULTAT PAR 2
TU SAURAS COMBIEN DE BOUGIES TU DOIS METTRE SUR TON GÂTEAU D'ANNIVERSAIRE

In this talk we will study projections of semi-martingales on various filtrations and give applications to some BSDEs.

A (special) **semimartingale** is a process X which admits a decomposition as $X = M + A$ where M is a (local) martingale and A a (predictable) process with bounded variation.

Let $\mathbb{F} \subset \mathbb{G}$ be two nested filtrations (i.e. $\mathcal{F}_t \subset \mathcal{G}_t$, $\forall t \geq 0$). The $\mathbb{F}$-(optional) projection of a $\mathbb{G}$-martingale $X^{\mathbb{G}}$ on $\mathbb{F}$ i.e., $X_t^{\mathbb{F}} = {}^oX_t^{\mathbb{G}} = \mathbb{E}(X_t^{\mathbb{G}} | \mathcal{F}_t), t \geq 0$ is an $\mathbb{F}$-martingale. The projection of a bounded variation process is not, in general a bounded variation process.

A simple case

The first step is to recall a well known lemma.

For any (bounded) process ϑ, let $\mathbb{F}$ be a filtration and define $\vartheta^{\mathbb{F}}_s = \mathbb{E}[\vartheta_s|\mathcal{F}_s]$. Then,

$$M_t := \mathbb{E}[\int_0^t \vartheta_u du|\mathcal{F}_t] - \int_0^t \mathbb{E}[\vartheta_u|\mathcal{F}_u]du = \mathbb{E}[\int_0^t \vartheta_u du|\mathcal{F}_t] - \int_0^t \vartheta^{\mathbb{F}}_u du$$

is an $\mathbb{F}$-martingale.

$$\begin{aligned}
\forall s < t, \quad \mathbb{E}[M_t|\mathcal{F}_s] &= \mathbb{E}\left[\mathbb{E}[\int_0^t \vartheta_u du|\mathcal{F}_t] - \int_0^t \mathbb{E}[\vartheta_u|\mathcal{F}_u]du|\mathcal{F}_s\right] \\
&= \mathbb{E}[\int_0^t \vartheta_u du|\mathcal{F}_s] - \mathbb{E}\left[\int_0^s \mathbb{E}[\vartheta_u|\mathcal{F}_u]du + \int_s^t \mathbb{E}[\vartheta_u|\mathcal{F}_u]du|\mathcal{F}_s\right] \\
= \mathbb{E}[\int_0^s \vartheta_u du|\mathcal{F}_s] &+ \mathbb{E}[\int_s^t \vartheta_u du|\mathcal{F}_s] - \int_0^s \mathbb{E}[\vartheta_u|\mathcal{F}_u]du - \mathbb{E}\left[\int_s^t \mathbb{E}[\vartheta_u|\mathcal{F}_u]du|\mathcal{F}_s\right] \\
&= M_s + \mathbb{E}[\int_s^t \vartheta_u du|\mathcal{F}_s] - \int_s^t \mathbb{E}[\vartheta_u|\mathcal{F}_s]du = M_s.
\end{aligned}$$

Let $\mathbb{F} \subset \mathbb{G}$ be two nested filtrations, ϑ be a $\mathbb{G}$-adapted integrable process and M be the $\mathbb{F}$-martingale

$$M_t = \mathbb{E}[\int_0^t \vartheta_s ds | \mathcal{F}_t] - \int_0^t \vartheta_s^{\mathbb{F}} ds \,.$$

The goal is to identify M in terms of the process ϑ and one (or more) specific $\mathbb{F}$-martingales which satisfy **predictable representation property** (PRP) on $\mathbb{F}$.

We recall that PRP is satisfied in a filtration $\mathbb{H}$ if there exists an $\mathbb{H}$-martingale (may be multidimensional) Y such that any $\mathbb{H}$ martingale X can be written as $X_t = X_0 + \int_0^t \varphi_s dY_s$ for a predictable process φ. Basic examples are Brownian filtration and filtration generated by a marked point process.

Assume that $\mathbb{F}$ is a Brownian filtration generated by W. In that case, predictable representation property (PRP) yields that there exists an $\mathbb{F}$-predictable process ψ such that $M_t = \int_0^t \psi_s dW_s$.

For any $\mathbb{F}$-adapted bounded process φ one has

$$\begin{aligned}
\mathbb{E}[\int_0^t \vartheta_s ds \int_0^t \varphi_s dW_s] &= \mathbb{E}[\mathbb{E}[\int_0^t \vartheta_s ds \,|\mathcal{F}_t] \int_0^t \varphi_s dW_s] \\
&= \mathbb{E}[\int_0^t \vartheta^{\mathbb{F}}_s ds \int_0^t \varphi_s dW_s] + \mathbb{E}[M_t \int_0^t \varphi_s dW_s]
\end{aligned}$$

hence,

$$\begin{aligned}
& \mathbb{E}[\int_0^t \vartheta_s ds \int_0^t \varphi_s dW_s] - \mathbb{E}[\int_0^t \vartheta^{\mathbb{F}}_s ds \int_0^t \varphi_s dW_s] \\
= \;& \mathbb{E}[M_t \int_0^t \varphi_s dW_s] = \mathbb{E}[\int_0^t \psi_s \varphi_s ds]
\end{aligned}$$

To proceed, we need to apply integration by parts applied to the product of $\mathbb{G}$-processes $\int_0^t \vartheta_s ds$ and $\int_0^t \varphi_s dW_s$ (if $\int_0^t \varphi_s dW_s$ is a $\mathbb{G}$-semimartingale!)

$$\mathbb{E}[\int_0^t \vartheta_s ds \int_0^t \varphi_s dW_s] = \mathbb{E}[\int_0^t \vartheta_s \left(\int_0^s \varphi_u dW_u\right) ds] + \mathbb{E}[\int_0^t \varphi_s \left(\int_0^s \vartheta_u du\right) dW_s] \,.$$

We now assume that there exists a $\mathbb{G}$-(predictable)process $a^{\mathbb{G}}$ such that W is a $\mathbb{G}$-semimartingale with decomposition

$$W_t = W_t^{\mathbb{G}} + \int_0^t a_s^{\mathbb{G}} ds$$

where $W^{\mathbb{G}}$ is a $\mathbb{G}$-Brownian motion, then $\int_0^t \varphi_s dW_s$ is a $\mathbb{G}$-semimartingale and

$$\begin{aligned}\mathbb{E}[\int_0^t \varphi_s \left(\int_0^s \vartheta_u du\right) dW_s] &= \mathbb{E}[\int_0^t \varphi_s \left(a_s^{\mathbb{G}} \int_0^s \vartheta_u du\right) ds] + \mathbb{E}[\int_0^t \varphi_s \left(\int_0^s \vartheta_u du\right) dW_s^{\mathbb{G}}] \\ &= \mathbb{E}[\int_0^t \varphi_s \left(a_s^{\mathbb{G}} \int_0^s \vartheta_u du\right) ds] \,.\end{aligned}$$

We have (by conditioning)

$$\mathbb{E}[\int_0^t \vartheta_s \left(\int_0^s \varphi_u dW_u\right) ds] = \mathbb{E}[\int_0^t \vartheta_s^{\mathbb{F}} \left(\int_0^s \varphi_u dW_u\right) ds] .$$

Then

$$\mathbb{E}[\int_0^t \vartheta_s ds \int_0^t \varphi_s dW_s] = \mathbb{E}[\int_0^t \vartheta_s^{\mathbb{F}} \left(\int_0^s \varphi_u dW_u\right) ds] + \mathbb{E}[\int_0^t \varphi_s a_s^{\mathbb{G}} \left(\int_0^s \vartheta_u du\right) ds]$$

With the same kind of computation

$$\begin{aligned}\mathbb{E}[\int_0^t \vartheta_s^{\mathbb{F}} ds \int_0^t \varphi_s dW_s] &= \mathbb{E}[\int_0^t \vartheta_s^{\mathbb{F}} \left(\int_0^s \varphi_u dW_u\right) ds] + \mathbb{E}[\int_0^t \varphi_s \left(\int_0^s \vartheta_u^{\mathbb{F}} du\right) dW_s] \\ &= \mathbb{E}[\int_0^t \vartheta_s^{\mathbb{F}} \left(\int_0^s \varphi_u dW_u\right) ds]\end{aligned}$$

and from

$$\mathbb{E}[\int_0^t \vartheta_s ds \int_0^t \varphi_s dW_s] = \mathbb{E}[\int_0^t \vartheta_s^{\mathbb{F}} \left(\int_0^s \varphi_u dW_u\right) ds] + \mathbb{E}[\int_0^t \varphi_s a_s^{\mathbb{G}} \left(\int_0^s \vartheta_u du\right) ds]$$

we get

$$\mathbb{E}[\int_0^t \psi_s \varphi_s ds] = \mathbb{E}[\int_0^t \varphi_s \, a_s^{\mathbb{G}} (\int_0^s \vartheta_u du) \, ds]$$

and this being true for any φ, this yields

$$\psi_s = \mathbb{E}[a_s^{\mathbb{G}} \int_0^s \vartheta_u du | \mathcal{F}_s].$$

As a check, if θ is $\mathbb{F}$-adapted, then $\psi_t = \mathbb{E}[a_t^{\mathbb{G}}|\mathcal{F}_t] \int_0^t \vartheta_u du \; = 0$ since $\mathbb{E}[a_t^{\mathbb{G}}|\mathcal{F}_t] = 0 \, , \forall s \geq 0$.

Note that this can be easily extended to the case where $\mathbb{F}$ admits a martingale (may be multi-dimensional or having jumps) which enjoys PRP.

Martingales

Let $\mathbb{F}$ be a Brownian filtration generated by W and $\mathbb{G}$ a bigger filtration $\mathbb{F} \subset \mathbb{G}$. Assume that

$$W_t = W_t^{\mathbb{G}} + \int_0^t a_s^{\mathbb{G}} ds$$

where $W^{\mathbb{G}}$ is a $\mathbb{G}$-Brownian motion. Then, if the $\mathbb{G}$-martingale $Y^{\mathbb{G}}$ admits the decomposition

$$Y_t^{\mathbb{G}} = Y_0^{\mathbb{G}} + \int_0^t y_s^{\mathbb{G}} dW_s^{\mathbb{G}}$$

where $Y_0^{\mathbb{G}}$ is a $\mathcal{G}_0$-measurable r.v. one has

$$E[Y_t^{\mathbb{G}} | \mathcal{F}_t] = Y_0^{\mathbb{G}} + \int_0^t \psi_s dW_s \,.$$

Computing

$$\mathbb{E}[Y_t^{\mathbb{G}} \int_0^t \varphi_s dW_s]$$

using, as before integration by parts, one finds

$$\psi_s = \mathbb{E}[y_s^{\mathbb{G}} + a_s^{\mathbb{G}} Y_s^{\mathbb{G}} | \mathcal{F}_s] \,.$$

Examples

These examples are known in enlargement of filtration theory.

• Initial enlargement

Let ζ be a random variable and consider $\mathbb{F}^{(\zeta)}$ the filtration $\mathcal{F}_t^{(\zeta)} = \mathcal{F}_t \vee \sigma(\zeta)$. Let η be the law of ζ. We say that ζ satisfies **Jacod's absolute continuity condition** if, for each $t \geq 0$, its conditional law is absolutely continuous with respect to its unconditional law, i.e.,

$$P_t(du) = p_t(u)\eta(du)$$

and $p(u)$ is an $\mathbb{F}$-martingale satisfying

$$\mathbb{E}[f(\zeta)|\mathcal{F}_t] = \int_{\mathbb{R}} f(u)p_{t-}(u)\eta(du)\,.$$

Suppose that ζ satisfies Jacod's absolute continuity condition.

Note that $p_t(\zeta) > 0$ and, in a Brownian filtration

$$dp_t(u) = \sigma_t(u) dW_t$$

hence, from results in enlargement of filtrations $a_t^{(\zeta)} = \frac{\sigma_t(\zeta)}{p_t(\zeta)}$. Furthermore, $W^{\mathbb{F}^{(\zeta)}}$ satisfies PRP.

Note that any $\mathcal{F}_t^{(\zeta)}$ random variable $Y_t^{\mathbb{F}^{(\zeta)}}$ can be written as $Y_t^{\mathbb{F}^{(\zeta)}} = Y_t^{\mathbb{F}}(\zeta)$ where $Y^{\mathbb{F}}(u)$ is $\mathbb{F}$-adapted.

Let $Y^{\mathbb{F}^{(\zeta)}}$ be the $\mathbb{F}^{(\zeta)}$-martingale $Y_t^{\mathbb{F}^{(\zeta)}} = \int_0^t y_s^{\mathbb{F}}(\zeta) dW_s^{\mathbb{F}^{(\zeta)}}$, then its $\mathbb{F}$-optional projection is $Y_t = \int_0^t \psi_s dW_s$ with

$$\psi_s = \int_{\mathbb{R}} (y_s^{\mathbb{F}}(u) p_s(u) + \sigma_s(u) Y_s^{\mathbb{F}}(u)\big) \eta(du)$$

- Progressive enlargement

Let τ be a positive r.v. satisfying Jacod's condition and $\mathbb{G}$ the progressive enlargement of $\mathbb{F}$ with τ, i.e. the smallest filtration which contains $\mathbb{F}$ and turns τ into a stopping time.

Then, any $\mathbb{F}$-martingale is a $\mathbb{G}$-semimartingale and there exists a pair of processes which enjoy PRP.

Shrinkage of a BSDE

Let us consider the following BSDE, in the initially enlarged filtration $\mathbb{G}$

$$dX_t^{\mathbb{G}} = -f(t, X_t^{\mathbb{G}}, Z_t^{\mathbb{G}})dt + Z_t^{\mathbb{G}} dW_t^{\mathbb{G}},\ X_T^{\mathbb{G}} = B$$

where $B \in \mathcal{G}_T$ is bounded, and $W^{\mathbb{G}}$ a $\mathbb{G}$-Brownian motion

$$W_t = W_t^{\mathbb{G}} + \int_0^t a_s^{\mathbb{G}} ds$$

and $f(s, x, z)$ is $\mathbb{F}$-adapted. Due to the PRP of $W^{\mathbb{G}}$ this BSDE admits a solution under the usual Lip. conditions on f. Recall that $Z_t^{\mathbb{G}} = Z_t^{\mathbb{F}}(\zeta)$ where $Z^{\mathbb{F}}(u)$ is $\mathbb{F}$-adapted.

Let $X_t = \mathbb{E}[X_t^{\mathbb{G}} | \mathcal{F}_t]$ and $Y_t^{\mathbb{G}} = \int_0^t Z_s^{\mathbb{G}} dW_s^{\mathbb{G}} = Y_t^{\mathbb{F}}(\zeta)$. One has

$$\mathbb{E}[Y_t^{\mathbb{G}} | \mathcal{F}_t] = \int_0^t \kappa_s dW_s$$

where

$$\kappa_s = \mathbb{E}[Z_s^{\mathbb{G}} + a_s^{\mathbb{G}} Y_s^{\mathbb{G}} | \mathcal{F}_s]\,.$$

Under Jacod's hypothesis, writing the density in absolute continuity assumption as

$$dp_t(u) = \sigma_t(u) dW_t$$

we know that $a_t^{\mathbb{G}} = \frac{\sigma_t(\zeta)}{p_t(\zeta)}$ and

$$\mathbb{E}[\int_0^t Z_s^{\mathbb{G}} dW_s^{\mathbb{G}} | \mathcal{F}_t) = \int_0^t \left(\int_{\mathbb{R}} \big(Z_s^{\mathbb{F}}(u) p_s(u) + Y_s^{\mathbb{F}}(u) \sigma_s(u)\big) \eta(du) \right) dW_s \,.$$

So that, setting $X_t = \mathbb{E}[X_t^{\mathbb{G}} | \mathcal{F}_t]$, and

$$f_t^{\mathbb{F}} = \mathbb{E}[f(t, X_t^{\mathbb{G}}, Z_t^{\mathbb{G}}) | \mathcal{F}_t] = \int_{\mathbb{R}} f(t, X_t^{\mathbb{F}}(u), Z_t^{\mathbb{F}}(u)) p_t(u) \eta(du)$$

we obtain

$$\begin{aligned} dX_t &= f_t^{\mathbb{F}} dt + \int_{\mathbb{R}} \big(Z_t^{\mathbb{F}}(u) p_t(u) + \sigma_t(u) Y_t^{\mathbb{F}}(u)\big) \eta(du)\, dW_t \\ &+ \int_{\mathbb{R}} \sigma_t(u) \int_0^t f(s, X_s^{\mathbb{F}}(u), Z_s^{\mathbb{F}}(u)) p_s(u) ds\, \eta(du))\, dW_t \,. \end{aligned}$$

In the particular case where $f(s,x,z) = \alpha_s + \beta_s x$, where α and β are $\mathbb{F}$-adapted

$$dX_t = -(\alpha_t + \beta_t X_t)dt + Z_t dW_t, \quad X_T = \mathbb{E}[B|\mathcal{F}_T]$$

with

$$\begin{aligned} Z_t &= \int_{\mathbb{R}} \sigma_t(u) \int_0^t f_s^{\mathbb{F}} ds \, \eta(du) \\ &+ \int_{\mathbb{R}} \big(Z_s(u) + \sigma_s(u) Y_s(u)\big) p_s(u)) \eta(du) \end{aligned}$$

Thank you for your attention

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AU LIEU DE VOUS EN FAIRE PARCE QUE VOUS AVEZ UN AN DE PLUS QUE L'ANNÉE PASSÉE
VOUS DEVRIEZ PLUTÔT VOUS RÉJOUIR D'EN AVOIR UN DE MOINS QUE L'ANNÉE PROCHAINE