Nonlinear random perturbations of PDEs.

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Joint work with:

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Freidlin and Koralov (*PTRF 2010*) have considered the following quasi-linear parabolic problem

$$\begin{cases} \partial_t u_\epsilon(t,x) = \frac{\epsilon}{2}\sum_{i,j=1}^d a_{i,j}(x, u_\epsilon(t,x))\,\partial_{ij} u_\epsilon(t,x) + \sum_{i=1}^d b_i(x)\,\partial_i u_\epsilon(t,x), \\ u_\epsilon(0,x) = g(x), \quad x \in \mathbb{R}^d, \end{cases} \tag{1}$$

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together with the randomly perturbed system where (B_t) is a d-dimensional Brownian motion and $a_{ij}(x,r) = (\sigma\sigma^\star)_{ij}(x,r)$.

$$\begin{cases} dX_\epsilon^{t,x}(s) = b(X_\epsilon^{t,x}(s))\, ds + \sqrt{\epsilon}\, \sigma(X_\epsilon^{t,x}(s), u_\epsilon(t-s, X_\epsilon^{t,x}(s)))\, dB_s, \\ X_\epsilon^{t,x}(0) = x, \end{cases} \tag{2}$$

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The PDE (1) and the SDE (2) are related by the relation:

$$u_\epsilon(r,x) = \mathbb{E} g(X_\epsilon^{r,x}(r)), \quad r \geq 0$$

The classical theory of (finite dimensional) parabolic, quasi-linear, PDEs guarantees that equation (2) admits a unique classical solution u_ϵ.

In their framework, Freidlin and Koralov complete a comprehensive program:

- Prove the Large Deviation Principle for the trajectories of $X_\epsilon^{t,x}$ and characterize the action functional.
- Study the exit problem for $(X_\epsilon^{t,x})$ from a fixed domain $D \subset \mathbb{R}^n$.
- Investigate the asymptotic behavior $\lim_{\epsilon\to 0} u_\epsilon(\lambda/\epsilon) := c(\lambda)$.

Motivation - Finite Dimensional Case

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Our objective is to start a similar program in the infinite-dimensional case (when equation (1) is a SPDE).

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Two (divergent) considerations:

- The asymptotic behavior as $\epsilon \to 0$ of the SDE (2) can be studied independently of the PDE (1).
- The PDE (1) is of independent interest.

We consider the randomly perturbed partial differential equation defined on a separable **Hilbert space** H, where ϵ is a small parameter.

$$\begin{cases} dX_\epsilon^{t,x}(s) = [AX_\epsilon^{t,x}(s) + b(X_\epsilon^{t,x}(s))]\; dt + \sqrt{\epsilon}\,\sigma(X_\epsilon^{t,x}(s), u_\epsilon(t-s, X_\epsilon^{t,x}(s)))\, dW_s, \\ X_\epsilon^{t,x}(0) = x \in H, \end{cases} \tag{3}$$

where u_ϵ satisfies (at least formally) the quasi-linear equations, $t \geq 0$

$$\begin{cases} D_t u_\epsilon(t,x) = \frac{\epsilon}{2}\mathrm{Tr}\left[\sigma\sigma^\star(x, u_\epsilon(t,x))D_x^2 u_\epsilon(t,x)\right] + \langle Ax + b(x), Du_\epsilon(t,x)\rangle_H, \\ u_\epsilon(0,x) = g(x), \quad x \in H. \end{cases} \tag{4}$$

and it holds

$$u_\epsilon(s,x) = \mathbb{E} g(X_\epsilon^{t,x}(t))$$

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Here

- $A : D(A) \subset H \to H$ is the generator of a strongly continuous, **Hilbert-Schmidt**, semigroup $(S(t))$ with:

$$\|S(t)\|_{\mathcal{L}_2(H)} \leq ce^{-\gamma t} \quad \text{for some } 0 < \gamma < 1/2 \text{ and all } t \in [0,1]$$

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- $g : H \to H$ is some non-linear Lipchitz mapping.

Notice that if

$$v_\epsilon^t(s,x) := u_\epsilon(t-s,x), \quad s\in[0,t]$$

then the above equations rewrite:

$$\begin{cases} D_t v_\epsilon^t(s,x) - \frac{\epsilon}{2}\mathrm{Tr}\left[\sigma\sigma^\star(x, v_\epsilon^t(s,x)) D_x^2 v_\epsilon^t(s,x)\right] + \langle Ax + b(x), Dv_\epsilon^t(s,x)\rangle_H = 0, \\ v_\epsilon^t(t,x) = g(x), \quad x\in H. \end{cases}$$

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and it holds

$$v_\epsilon^t(s,x) = \mathbb{E} g(X_\epsilon^{t-s,x}(t-s))$$

thus by markovianity

$$v_\epsilon^t(s, X_\epsilon^{t,x}(s)) = \mathbb{E}\left(g(X_\epsilon^{t,x}(t) \middle| \mathcal{F}_s^W \right)$$

The equations for $X_\epsilon^{t,x}$ can be rewritten in a closed form as:

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The well posedness of the above equation **for small** ϵ can be easily established:

Theorem

Fix $T > 0$ there exists $\bar{\epsilon}(T) > 0$ such that for all $t \leq T$ and $\epsilon \leq \bar{\epsilon}(T)$ there exists a unique solution $(X_\epsilon^{t,x})(s)_{s\in[0,t]})$ with continuous trajectories. Moreover

$$\mathbb{E}(\sup_{s\in[0,t]} |X_\epsilon^{t,x}(s)|^2) \leq C(T)(1+|x|^2)$$

$$\mathbb{E}(\sup_{s\in[0,t]} |X_\epsilon^{t,x}(s) - X_\epsilon^{t,x'}(s)|^2) \leq C(T)|x-x'|^2$$

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Remark: if A is dissipative then $\bar{\epsilon} > 0$ can be chosen independently on T

When $\epsilon \to 0$ the laws of $(X_\epsilon^{t,x})$ converge to the Dirac measure centerd in in Z^x where

$$\frac{d}{ds} Z^x(s) = AZ^x(s) + b(Z^x(s)); \qquad Z_\epsilon^x(0) = x$$

The events $\Gamma \subset C([0,t];H)$ that do not contain Z^x describe a **deviant** behavior.

We want to know how deviant a particular event Γ such that $Z^x \notin \bar{\Gamma}$ is, more precisely we want to compute the exponential rate at which $\mathcal{L}(X_\epsilon^{t,x})(\Gamma)$ goes to zero.

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A family of probability measures $\{\mu_\epsilon\}_{\epsilon>0}$ on a Polish space E satisfies a large deviation principle, with speed $1/\epsilon$ and **action functional** $I : E \to [0, +\infty]$ if

- for every $A \subset E$ open $\liminf_{\epsilon \to 0} \epsilon \log \mu_\epsilon(A) \geq - \inf_{x \in A} I(x)$,
- for every $C \subset E$ closed $\limsup_{\epsilon \to 0} \epsilon \log \mu_\epsilon(C) \leq - \inf_{x \in C} I(x)$.
- for every $s \geq 0$, the set $\{I(x) \leq s\}$ is compact in E,

Theorem (S. Cerrai, G. Guatteri, G.T.)

Assume that $(S(t))$ is an analytic semigroup then the family $\{\mathcal{L}(X_\epsilon^{t,x})\}_{\epsilon\in(0,\bar{\epsilon})}$ satisfies a ***LDP*** *in $C([0,t];H)$ governed by the action functional*

$$I_{t,x}(f) = \frac{1}{2}\inf\left\{\int_0^t \|\varphi(s)\|_H^2\,ds \,:\, f(s) = X_\varphi^{t,x}(s),\ s\in[0,t]\right\},$$

where f ranges over continuous function $[0,t]\to H$ with $f(0)=x$ and $X_\varphi^{t,x}$ is the unique mild solution of problem

$$\begin{cases} (X_\varphi^{t,x}(s))' = AX_\varphi^{t,x}(s) + b(X_\varphi^{t,x}(s)) + \sigma\left(X_\varphi^{t,x}(s), g(Z^{X_\varphi^{t,x}(s)}(t-s))\right)\varphi(s) \\ X_\varphi^{t,x}(0) = x\in H, \end{cases}$$

We recall that for every $y\in H$, (Z^y) verifies:

$$Z^y(s) = e^{sA}y + \int_0^s e^{(s-r)A}b(Z^y(r))\,dr.$$

By general results, [A. Budhiraja, P. Dupuis, V. Maroulas, *Ann. Probab.* 2008] **a LDP with rate function** $I_{t,x}(f) = \frac{1}{2} \inf \left\{ \int_0^t \|\varphi(s)\|_H^2 \, ds \; : \; f = X_\varphi^{t,x} \right\}$, **holds** when the conditions below are verified

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- for every $t, R > 0$, the level sets $\{I_{t,x} \leq R\}$ are compact in $C([0,t];H)$.
- For every $M > 0$ denote by $\Lambda_{t,M}$ the set of progressively measurable processes φ such that $\|\varphi\|_{L^2(0,t;H)} \leq M$, $\mathbb{P}-$a.s..

  For all $\{\varphi_\epsilon\}_{\epsilon>0} \subseteq \Lambda_{t,M}$ and $\varphi \in \Lambda_{t,M}$:

  if $\varphi_\epsilon \rightharpoonup \varphi$ **weakly in** $L^2(0,t;H)$ **in distribution**

  then $X_{\varphi_\epsilon,\epsilon}^{t,x} \to X_\varphi^{t,x}$ **strongly in** $C([0,t];H)$ **in distribution**

Where $X_{\varphi_\epsilon,\epsilon}^{t,x}$ solves:

$$\begin{cases} dX_{\varphi_\epsilon,\epsilon}^{t,x}(s) = \left[AX_{\varphi_\epsilon,\epsilon}^{t,x}(s) + b((X_{\varphi_\epsilon,\epsilon}^{t,x}(s)) \right] ds + \\ \qquad \sqrt{\epsilon}\,\sigma\left(X_{\varphi_\epsilon,\epsilon}^{t,x}(s), v_\epsilon^t(s, X_{\varphi_\epsilon,\epsilon}^{t,x}(s))\right) dW_s + \sigma\left(X_{\varphi_\epsilon,\epsilon}^{t,x}(s), v_\epsilon^t(s, X_{\varphi_\epsilon,\epsilon}^{t,x}(s))\right) \varphi_\epsilon(s) ds, \\ X_{\varphi_\epsilon,\epsilon}^{t,x}(0) = x \in H, \end{cases}$$

and $v_\epsilon^t(s,x) = \mathbb{E} g(X_\epsilon^{t-s,x}(t-s))$

We will concentrate on the second condition.
It is worth noticing that the controlled SPDE

$$\begin{cases} dX^{t,x}_{\varphi_\epsilon,\epsilon}(s) = \left[AX^{t,x}_{\varphi_\epsilon,\epsilon}(s) + b((X^{t,x}_{\varphi_\epsilon,\epsilon}(s))\right] ds + \\ \qquad \sqrt{\epsilon}\,\sigma\left(X^{t,x}_{\varphi_\epsilon,\epsilon}(s), v^t_\epsilon(s, X^{t,x}_{\varphi_\epsilon,\epsilon}(s))\right) dW_s + \sigma\left(X^{t,x}_{\varphi_\epsilon,\epsilon}(s), v^t_\epsilon(s, X^{t,x}_{\varphi_\epsilon,\epsilon}(s))\right) \varphi_\epsilon(s) ds, \\ X^{t,x}_{\varphi_\epsilon,\epsilon}(0) = x \in H, \end{cases}$$

can be rewritten as

$$\begin{cases} dX^{t,x}_{\varphi_\epsilon,\epsilon}(s) = \left[AX^{t,x}_{\varphi_\epsilon,\epsilon}(s) + b((X^{t,x}_{\varphi_\epsilon,\epsilon}(s))\right] ds + \\ \qquad \sqrt{\epsilon}\,\sigma\left(X^{t,x}_{\varphi_\epsilon,\epsilon}(s), \mathbb{E}^{\mathbb{P}^\epsilon}\left(g(X^{t,x}_{\varphi_\epsilon,\epsilon})(t) \middle| \mathcal{F}^W_s\right)\right) dW^\epsilon_s, \\ X^{t,x}_\epsilon(0) = x \in H, \end{cases}$$

where $W^\epsilon_s := W_s + \frac{1}{\sqrt{\epsilon}} \int_0^s \varphi(r) dr$
and $\mathbb{P}^\epsilon$ is the probability under which (W^ϵ) is a Wiener process.

We have to prove that for all $\{\varphi_\epsilon\}_{\epsilon>0}$ with $|\varphi_\epsilon|_{L^2(0,t;H)} \leq M$, $\mathbb{P}$-a.s.
if $\varphi_\epsilon \rightharpoonup \varphi$ weakly in $L^2(0,t;H)$ in distribution then $X^{t,x}_{\varphi_\epsilon,\epsilon} \to X^{t,x}_{\varphi}$ strongly in $C([0,t];H)$ in distribution.

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We proceed as follows:

- Use Skorohod's Theorem to pass from convergence in law to $\mathbb{P}$-a.s. (weak) convergence $\varphi_\epsilon \rightharpoonup \phi$ in $L^2(0,t;H)$

Sketch of the proof

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- prove that for some $\delta > 0$, $\alpha > 0$

$$\|X^{t,x}_{\varphi_\epsilon} - X^{t,x}_{\varphi}\|_{C^\delta([0,t];\mathcal{D}((-A)^\alpha))} \leq \|\varphi_\epsilon - \varphi\|_{L^2(0,t;H)}$$

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  where $C^\delta([0,t];\mathcal{D}((-A)^\alpha))$ is the space of δ-Holder continuous functions with values in $\mathcal{D}((-A)^\alpha)$
- exploit the compact embedding $C^\delta([0,t];\mathcal{D}((-A)^\alpha)) \hookrightarrow C([0,t];H)$ to obtain strong convergence of $X^{t,x}_{\varphi_\epsilon,\epsilon} \to X^{t,x}_{\varphi}$ in $C([0,t];H)$.

The proof is completed. □

We come back to the PDE in infinite variables

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We are now interested into "classical" solutions. We will only able to consider the case when there exist a bounded non-negative symmetric operator Q, a continuous mapping $f : H \times \mathbb{R} \to \mathcal{L}_1^+(H)$ and $\delta > 0$ **small enough** such that

$$\sigma^\star\sigma(x,r) = Q + \delta f(x,r), \qquad x \in H, \quad r \in \mathbb{R}.$$

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We are now interested into "classical" solutions. We will only able to consider the case when there exist a bounded non-negative symmetric operator Q, a continuous mapping $f : H \times \mathbb{R} \to \mathcal{L}_1^+(H)$ and $\delta > 0$ **small enough** such that

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On A and Q we assume:

- The semigroup (S) generated by A is of negative type
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- If we define $\Lambda_t := Q_t^{-1/2}S(t)$ there exists some $\lambda > 0$ such that $\|\Lambda_t\|_{\mathcal{L}(H)} \leq c\,(t\wedge 1)^{-1/2}e^{-\lambda t}, \qquad t > 0.$

An example

Let $H = L^2(\mathcal{O})$, for some bounded interval $\mathcal{O} \subset \mathbb{R}$, and let $\{e_i\}_{i \in \mathbb{N}}$ be an orthonormal basis of H contained in $L^\infty(\mathcal{O})$.

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- A is the realization of the Laplace operator with Dirichlet boundary conditions in $\mathcal{O}$
- $Q = I$.

We fix non-negative numbers $\{\lambda_i\}_{i \in \mathbb{N}}$, and we assume that $\sum_{i=1}^{\infty} \lambda_i \|e_i\|_{L^\infty(\mathcal{O})} < \infty$.

- For every $x \in H$, $r \in \mathbb{R}$, and $i \in \mathbb{N}$, we define

  $$[f(x,r)e_i](\xi) = \mathfrak{f}_i(x(\xi), r)\lambda_i e_i(\xi), \qquad \xi \in \mathcal{O},$$

  for some nice smooth functions $\mathfrak{f}_i : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

We rewrite the quasi-linear equation as a **perturbation** of Ornstein-Uhlenbeck (ϵ is now irrelevant since we have δ small; we set $\epsilon = 1$ and simplify notation)

$$\begin{cases} D_s u(s,x) = \mathcal{L}u(s,x) + \frac{\delta}{2}\mathrm{Tr}\left[f(x,u(s,x))D_x^2 u(s,x)\right] + \langle b(x), Du(s,x)\rangle_H, \\ u(0,x) = g(x), \quad x \in H, \end{cases}$$

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Theorem (S. Cerrai, G. Guatteri, G.T., *JFA* 2024)

Fix $\frac{1}{2} < \eta < 1$, $0 < \vartheta < \frac{\eta-1}{2}$, let $\varrho = \frac{1-\eta+\vartheta}{2}$ and assume that $g \in C_b^\eta(H)$. There exists $\bar{\delta} > 0$ such that for every $\delta \leq \bar{\delta}$, $t > 0$ there exists a unique classical solution $u \in C([0,t], H)$. Moreover denoting Hölder norms by $\|\cdot\|_\alpha$.

$$\sup_{s\in(0,t]}\left(\|u(s,\cdot)\|_\eta + (s\wedge 1)^\varrho\|D_x u(s,\cdot)\|_\vartheta + (s\wedge 1)^{\varrho+\frac{1}{2}}\|D_x^2 u(s,\cdot)\|_\vartheta\right) \leq c_\delta\,\|g\|_\eta,$$

for some constant $c_\delta > 0$ independent of t.

Step 1.

We write the quasi-linear PDE in mild form as

$$u(s,x) = R_s g(x) + \frac{\delta}{2}\int_0^s R_{s-r}\mathrm{Tr}\left[F(u(r,\cdot))D_x^2 u(r,\cdot)\right](x)\,dr + \int_0^s R_{s-r}\langle b(\cdot), Du(r,\cdot)\rangle_H(x)\,dr.$$

where $F(\psi)(x) = f(x,\psi(x))$ and

$$R_s\psi := \int_H \psi(S(s)x+y)\mathcal{N}_{Q_s}(dy), \qquad \psi \in B_b(H)$$

is the Ornstein-Uhlenbeck semigroup.

Notice that if $\xi(s,x) = [R_s\psi](x)$ then ξ is a classical solution of the linear PDE in H:

$$\begin{cases} \frac{d}{ds}\xi(s,x) = \mathcal{L}\,\xi(s,x) \\ \xi(0,x) = \psi(x) \end{cases}$$

By [Da Prato, Zabczyk 2002] we know that the operator R_s, $s > 0$ is smoothing. For instance for every $0 \leq \beta \leq \alpha$ there exist some $c_{\alpha,\beta} > 0$ such that

$$\|R_s\varphi\|_\alpha \leq c_{\alpha,\beta}\,(s \wedge 1)^{-\frac{\alpha-\beta}{2}}\,\|\varphi\|_\beta, \qquad t > 0.$$

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Moreover R_s, $s > 0$ maps $C^0(H)$ into $C^\infty(H)$. In particular if we set

$$\|\varphi\|_{s,\theta} := \left(\|\varphi\|_0 + e^{-\omega\theta s}\,[\varphi]_\theta\right)$$

where $[\varphi]_\theta$ is the θ-Hölder seminorm then il holds for every $n \in \mathbb{N}$ and $0 \leq \theta \leq \rho \leq 1$

$$\|D^n R_s\varphi\|_\theta \leq c_{n,\theta,\rho}\,(s \wedge 1)^{-\frac{n-(\rho-\theta)}{2}}\,e^{-\lambda n s}\|\varphi\|_{s,\rho}, \qquad s > 0.$$

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Step 2: We try to establish a contraction argument, we have to take into account that:

- we deal with a second order nonlinear term $\mathrm{Tr}\left[F(u(r,\cdot))D_x^2 u(r,\cdot)\right]$ in our function space we have to go up to **second order differentiability**.
- g is not smooth we have to cope with **explosions** of norms when $s \searrow 0$.

Thus we chose to work with the space of smooth Hölder continuous functions in H endowed with a slight modification of the norm:

$$\sup_{s\in(0,t]}\left(\|u(s,\cdot)\|_\eta + (s\wedge 1)^\varrho \|D_x u(s,\cdot)\|_\vartheta + (s\wedge 1)^{\varrho+\frac{1}{2}}\|D_x^2 u(s,\cdot)\|_\vartheta\right) := |||u|||_{\eta,\rho,\theta,t}$$

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We prove, exploiting smoothing estimates for (R), that if

$$\Gamma[u](s,x) := R_s g(x) + \frac{\delta}{2}\int_0^s R_{s-r}\mathrm{Tr}\left[F(u(r,\cdot))D_x^2 u(r,\cdot)\right](x)\,dr + \int_0^s R_{s-r}\langle b(\cdot), Du(r,\cdot)\rangle_H(x)\,dr.$$

and **if δ and t are sufficiently small** then Γ is a contraction with respect to norm $|\!|\!|\cdot|\!|\!|_{\eta,\rho,\theta,t}$. Thus **a unique local mild solution exists.**

Step 3: We show, again exploiting the smoothing of R_s, that any local mild solution u is in fact a **classical** solution. In particular,

- $u(s,\cdot) \in C_b^2(H)$, for every $s \in (0,t]$,
- $QD_x^2 u(t,x) \in \mathcal{L}_1(H)$ (is a trace-class operator),
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Step 4: We show that local solution satisfies an *a-priori* bound, that is lies in a ball with respect to the norm $|||u|||_{\eta,\rho,\theta,t}$.

Notice that the estimate if the C^0 part of the norm, that is the **maximum principle**

$$\|u(s,\cdot)\|_0 \le \|g\|_0$$

comes form stochastic representation of the solution u.

We conclude **global existence** by a standard iterative process. □

We remark that, even in the case in which f only depends on x, that is for equation:

$$\begin{cases} D_t u(t,x) = \mathcal{L}u(t,x) + \mathrm{Tr}\left[\mathrm{f}(x) D_x^2 u(t,x)\right] + \langle b(x), Du(t,x)\rangle_H, \\ u(0,x) = g(x), \quad x \in H, \end{cases} \tag{6}$$

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In particular, the regularity estimates for u, obtained in the above papers, depending on the regularity of f, are not good enough (for us).
Namely we can not attack our PDE by a contraction argument following the schema:

$$u \mapsto f \circ u := \mathrm{f} \mapsto u$$

where the last map is indeed given by equation (6).

Some References

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Thank You

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Happy Birthday Ying!