The geometic approach to the extendability problem for linear codes

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Abstract. We introduce a special class of arcs in $\mathrm{PG}(r,q)$ called $(t \mod q)$-arcs. These are geometric objects whose structure is related to the extendability of linear codes. We present constructions and general structure results for $(t \mod q)$-arcs. Based on the characterization of $(3 \mod 5)$-arcs in $\mathrm{PG}(2,5)$ and $\mathrm{PG}(3,5)$, we prove the nonexistence of certain Griesmer codes over $\mathbb{F}_5$.

Keywords: finite projective geometries, arcs, blocking sets, divisible arcs, quasi-divisible arcs, Griesmer bound, $(t \mod q)$-arcs, extendable arcs, extendable linear codes

1 Introduction

In this paper we study the connection between the extendability of arcs in finite projective geometries (resp. the extendability of linear codes over finite fields) and their divisibility properties. A linear code over $\mathbb{F}_q$ is said to be divisible with divisor $\Delta > 1$ if the weight of every codeword is a multiple of Δ. It is almost straightforward that if $(\Delta, q) = 1$ then a code of full length (a code in which no coordinate is identically zero) is a Δ-fold replicated code, i.e. a concatenation of Δ identical codes [19].

Griesmer codes of minimum weight divisible by the characteristic of the ground field also have divisibility properties. We quote here the remarkable result by H. N. Ward from [19].

Theorem 1. *Let C be a Griesmer code over $\mathbb{F}_p$, p a prime. Then if p^e divides the minimum weight of C, then p^e is a divisor of the code.*

A linear $[n,k,d]_q$-code is said to be t-quasidivisible modulo Δ if $d \equiv -t \pmod{\Delta}$ and all weights in the code are congruent to $-t,\ldots,-1,0$ modulo Δ. Codes obtained by t-fold puncturing of a divisible code with divisor Δ are t-quasidivisible modulo Δ. It happens very often, especially for small values of t, that t-quasidivisible codes are t-extendable to a divisible code. For instance, the classical theorem by Hill and Lizak [7,8] says that every linear $[n,k,d]$-code with weights 0 and d modulo q, where $(d,q)=1$, is extendable to a $[n+1,k,d+1]_q$-code. Here the most common case is $d \equiv -1 \pmod{q}$. Using the notion of a quasidivisible code, this is equivalent to saying that every 1-quasidivisible code is extendable. Recently Maruta produced a lot of extendability results of this type [14,15,16,17,20]. The most interesting of them says that if an $[n,k,d]_q$-code with $q \geq 5$ odd, and $d \equiv -2 \pmod{q}$ has only weights $-2,-1,0 \pmod{q}$ then it is extendable [16]. This is equivalent to the statement that every 2-quasidivisible code over a field of order $q \geq 5$, q odd, is extendable.

An attempt for a unified approach to the question of code extendability was made in [12], where the problem was tackled from its geometric side. It is well-known that linear codes over finite fields and arcs in the finite geometries $\mathrm{PG}(k-1,q)$ are equivalent objects: with every linear $[n,k,d]_q$-code C one can associate an $(n,n-d)$-arc $\mathcal{K}_C$ in $\mathrm{PG}(k-1,q)$ (clearly, in a non-unique way) so that two codes C_1 and C_2 are isomorphic if and only if the arcs $\mathcal{K}_{C_1}$ and $\mathcal{K}_{C_2}$ associated with them are projectively equivalent [3,10,18]. Arcs associated with codes meeting the Griesmer bound are called Griesmer arcs.

In this paper, we introduce $(t \mod q)$-arcs in $\mathrm{PG}(r,q)$. These are geometric objects whose structure is related to the extendability of certain arcs and codes. We present structure results on $(t \mod q)$-arcs and prove the nonexistence of certain Griesmer codes over $\mathbb{F}_5$.

2 Quasidivisible Arcs and Extendability

In order to fix the notation, we introduce some basic definitions and facts on multisets. Consider the geometry $\Sigma = \mathrm{PG}(r,q)$, $r \geq 2$, $q = p^h$. Denote by $\mathcal{P}$ be the set of points and by $\mathcal{H}$ the set of hyperplanes of Σ. Every mapping $\mathcal{K} : \mathcal{P} \to \mathbb{N}_0$ from the pointset of the geometry to the non-negative integers is called a multiset in Σ. This mapping is extended additively to every subset $\mathcal{Q}$ of $\mathcal{P}$ by $\mathcal{K}(\mathcal{Q}) = \sum_{P\in\mathcal{Q}} \mathcal{K}(P)$. The integer $n := \mathcal{K}(\mathcal{P})$ is called the cardinality of $\mathcal{K}$. The support of $\mathcal{K}$ is the set of all points of positive multiplicity:

$$\operatorname{supp}\mathcal{K} = \{P \in \mathcal{P} | \mathcal{K}(P) > 0\}.$$

Multisets with $\mathcal{K}(P) \in \{0,1\}$ are called projective. They can be viewed as sets by identifying them with their support. For every set of points $\mathcal{Q} \subset \mathcal{P}$ we define its characteristic (multi)set $\chi_{\mathcal{Q}}$ by

$$\chi_{\mathcal{Q}}(P) = \begin{cases} 1 & \text{if } P \in \mathcal{Q}, \\ 0 & \text{otherwise.} \end{cases}$$

Denote by a_i the number of hyperplanes H of Σ with $\mathcal{K}(H) = i$. The sequence (a_i) is called the spectrum of $\mathcal{K}$.

Multisets can be viewed as arcs or blocking sets. A multiset $\mathcal{K}$ in Σ is called an (n, w)-multiarc (or simply (n, w)-arc) if (1) $\mathcal{K}(\mathcal{P}) = n$, (2) $\mathcal{K}(H) \le w$ for every hyperplane H, and (3) there exists a hyperplane H_0 with $\mathcal{K}(H_0) = w$. Similarly, a multiset $\mathcal{K}$ in Σ is called an (n, w)-blocking set with respect to the hyperplanes (or (n, w)-minihyper) if (1) $\mathcal{K}(\mathcal{P}) = n$, (2) $\mathcal{K}(H) \ge w$ for every hyperplane H, and (3) there exists a hyperplane H_0 with $\mathcal{K}(H_0) = w$.

An (n, w)-arc $\mathcal{K}$ in Σ is called t-extendable, if there exists an $(n + t, w)$-arc $\mathcal{K}'$ in the same geometry with $\mathcal{K}'(P) \ge \mathcal{K}(P)$ for every point $P \in \mathcal{P}$. An arc is called simply extendable if it is 1-extendable. Extendable arcs are associated with extendable codes and vice versa. Similarly, an (n, w)-blocking set $\mathcal{K}$ in Σ is called t-reducible, if there exists an $(n - t, w)$-blocking set $\mathcal{K}'$ in Σ with $\mathcal{K}'(P) \le \mathcal{K}(P)$ for every point $P \in \mathcal{P}$.

An (n, w)-arc $\mathcal{K}$ with spectrum (a_i) is said to be divisible with divisor $\Delta > 1$ if $a_i = 0$ for all $i \not\equiv n \pmod{\Delta}$. The (n, w)-arc $\mathcal{K}$ with $w \equiv n + t \pmod{q}$ is called t-quasidivisible with divisor $\Delta > 1$ (or t-quasidivisible modulo Δ) if $a_i = 0$ for all $i \not\equiv n, n+1, \ldots, n+t \pmod{\Delta}$, $1 \le t \le q - 1$. It is easily seen that linear codes associated with divisible (resp. t-quasidivisible) arcs are divisible (resp. t-quasidivisible) with the same divisor. The divisors Δ in this paper are always powers of the characteristic of the base field $\mathbb{F}_q$.

Given the projective geometry $\Sigma = \mathrm{PG}(r, q)$, we define its dual $\widetilde{\Sigma}$ in the usual way: we take the hyperplanes of Σ as points, the subspaces of codimension two as lines, and preserve the incidence. If S is a subspace of Σ of (projective) dimension s, we shall denote by $\widetilde{S}$ the subspace in $\widetilde{\Sigma}$ associated with S. Clearly, the dimension of $\widetilde{S}$ in $\widetilde{\Sigma}$ is $r - 1 - s$. Now for every t-quasidivisible (n, w)-arc $\mathcal{K}$ with divisor q in Σ, $t < q$, we can define a dual arc $\widetilde{\mathcal{K}}$ in the geometry $\widetilde{\Sigma}$ by

$$\widetilde{\mathcal{K}} : \begin{cases} \widetilde{\mathcal{P}} \to \{0, 1, \ldots, t\} \\ \widetilde{H} \to \widetilde{\mathcal{K}}(\widetilde{H}) = n + t - \mathcal{K}(H) \pmod{q} \end{cases}, \tag{1}$$

where $\widetilde{\mathcal{P}}$ is the set of all points in $\widetilde{\Sigma}$, i.e. the set of hyperplanes in Σ. This means that hyperplanes of multiplicity congruent to $n + a \pmod{q}$ become $(t - a)$-points in the dual geometry. In particular, maximal hyperplanes are 0-points with respect to $\widetilde{\mathcal{K}}$. Let us note that the size of $\widetilde{\mathcal{K}}$ depends on the spectrum of $\mathcal{K}$ and not just on the parameters of the arc. The following simple result establishes the basic divisibility properties of $\widetilde{\mathcal{K}}$.

Theorem 2. *[12] Let $\mathcal{K}$ be an (n, w)-arc in $\Sigma = \mathrm{PG}(r, q)$ which is t-quasidivisible modulo q with $t < q$. For every subspace $\widetilde{S}$ of $\widetilde{\Sigma}$, with $\dim \widetilde{S} \geq 1$,*

$$\widetilde{\mathcal{K}}(\widetilde{S}) \equiv t \pmod{q}.$$

The above observation justifies the following definition. Let t be a fixed non-negative integer. An arc $\mathcal{F}$ in Σ is called a $(t \mod q)$-arc if for every subspace S of dimension at least 1, $\mathcal{F}(S) \equiv t \pmod{q}$. By Theorem 2 if $\mathcal{K}$ is an arc which is t-quasidivisible modulo q then its dual $\widetilde{\mathcal{K}}$, given by (1), is a $(t \mod q)$-arc with point multiplicities that do not exceed t. Let $\mathcal{K}$ be an (n, w)-arc in Σ which is t-quasidivisible modulo q. The following theorem connects the extendability of $\mathcal{K}$ with the structure of the $(t \mod q)$-arc $\widetilde{\mathcal{K}}$.

Theorem 3. *[12] Let $\mathcal{K}$ be an (n, w)-arc in $\Sigma = \mathrm{PG}(r, q)$ which is t-quasidivisible modulo q with $t < q$, and let its dual $\widetilde{\mathcal{K}}$ be defined by (1). If*

$$\widetilde{\mathcal{K}} = \sum_{i=1}^{c} \chi_{\widetilde{P}_i} + \mathcal{K}'$$

for some multiset $\mathcal{K}'$ in $\widetilde{\Sigma}$ and c not necessarily different hyperplanes $\widetilde{P}_1, \ldots, \widetilde{P}_c$ in $\widetilde{\Sigma}$, then $\mathcal{K}$ is c-extendable. In particular, if $\widetilde{\mathcal{K}}$ contains a hyperplane in its support, then $\mathcal{K}$ is extendable.

3 Structure Results for $(t \mod q)$-Arcs

In this section, we study $(t \mod q)$-arcs as a purely geometric object without relation to the extendability problem. We start with a straightforward construction.

Theorem 4. *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be a $(t_1 \mod q)$- and a $(t_2 \mod q)$-arc, respectively, in $\mathrm{PG}(r, q)$. Then $\mathcal{F}_1 + \mathcal{F}_2$ is a $(t \mod q)$-arc with $t = t_1 + t_2 \pmod{q}$. Similarly, $\alpha\mathcal{F}_1$, $\alpha \in \{0, \ldots, p-1\}$ is a $(t \mod q)$ arc with $t \equiv \alpha t_1 \pmod{q}$.*

This theorem has a nice corollary for the case $t = 0$ when $q = p$ is a prime.

Corollary 1. *Let $\mathcal{F}$ and $\mathcal{G}$ be $(0 \mod p)$-arcs in $\mathrm{PG}(r, p)$, where p is a prime. Then $\mathcal{F} + \mathcal{G}$ and $\alpha\mathcal{F}$, $\alpha \in \{0, \ldots, p-1\}$, are also $(0 \mod p)$-arcs. In particular, the set of all $(0 \mod p)$-arcs in $\mathrm{PG}(r, p)$ is a vector space over $\mathbb{F}_p$.*

The next construction is less obvious.

Theorem 5. *Let $\mathcal{F}_0$ be a $(t \mod q)$-arc in a hyperplane $H \cong \mathrm{PG}(r-1,q)$ of $\Sigma = PG(r,q)$. For a fixed point $P \in \Sigma \setminus H$, define the arc $\mathcal{F}$ in Σ as follows:*

– $\mathcal{F}(P) = t$;

– for each point $Q \neq P$: $\mathcal{F}(Q) = \mathcal{F}_0(R)$ where $R = \langle P, Q\rangle \cap H$.

Then the arc $\mathcal{F}$ is a $(t \mod q)$-arc of size $q|\mathcal{F}_0| + t$.

We call the $(t \mod q)$-arc obtained by Theorem 5 a *lifted arc from* $\mathcal{F}_0$ and the point P – *lifting point*. We can generalize slightly the notion of a lifted arc replacing the point P by a subspace U. Let $\mathcal{F}_0$ be a $(t \mod q)$-arc in the subspace V of $\Sigma = \mathrm{PG}(r,q)$ and let U be a complementary subspace in Σ with $\dim U + \dim V = r - 1$, $U \cap V = \varnothing$. The arc $\mathcal{F}$ in Σ defined by

– $\mathcal{F}(P) = t$ for every point $P \in U$;

– for each point $Q \neq P$: $\mathcal{F}(Q) = \mathcal{F}_0(R)$ where $R = \langle U, Q\rangle \cap H$

is called an arc *lifted from the subspace* U. Obviously $\mathcal{F}$ is also a $(t \mod q)$-arc. Let us note that if an arc is lifted from a subspace then it can be considered as lifted from any point of that subspace. We have also a partial converse of this observation.

Lemma 1. *Let $\mathcal{F}$ be a $(t \mod q)$-arc in $\mathrm{PG}(r,q)$ which is lifted from the points P and Q, $P \neq Q$. Then $\mathcal{F}$ is also lifted from the line PQ. In particular, the lifting points of a $(t \mod q)$-arc form a subspace S of $\mathrm{PG}(r,q)$.*

From this point on, we consider only geometries $\Sigma = \mathrm{PG}(r,p)$ over prime fields $\mathbb{F}_p$. Denote by V the set of all $(0 \mod p)$ arcs in $\mathrm{PG}(r,p)$. The following observation is similar to the one made in Theorem 4 and Corollary 1.

Lemma 2. *Let $\mathcal{F}$ and $\mathcal{G}$ be $(0 \mod p)$-arcs in $\mathrm{PG}(r,p)$ that are lifted from the same subspace U, $\dim U \geq 0$. Then $\mathcal{F}+\mathcal{G}$ and $\alpha\mathcal{F}$, $\alpha \in \{0,1,\dots,p-1\}$, are also $(0 \mod p)$-arcs lifted from U. In particular, the $(0 \mod p)$-arcs lifted from U form a subspace of V.*

Now let us denote by A the points-by-lines incidence matrix of $\mathrm{PG}(r,p)$, p prime, for some fixed order of the points. Let $\mathcal{F}$ be an arc in $\mathrm{PG}(r,p)$ with point multiplicities not exceeding $p-1$. Then the arc $\mathcal{F}$ is represented by a vector $\boldsymbol{x}$ over $\mathbb{F}_p$:

$$\boldsymbol{x} = (\mathcal{F}(P_1), \dots, \mathcal{F}(P_{\frac{p^{r+1}-1}{p-1}})),$$

where the point multiplicities are viewed as elements of $\mathbb{F}_p$. It is obvious that $\mathcal{F}$ is a $(0 \mod p)$-arc if and only if

$$\boldsymbol{x}A = \mathbf{0}, \tag{2}$$

where $\mathbf{0}$ is the zero vector. Hence

$$\dim V = \frac{q^{r+1}-1}{q-1} - \mathrm{rk}_p\, A.$$

The rank of A is known from the celebrated theorem by Hamada [6] which is stated below in its general form.

Theorem 6. *The rank over $\mathbb{F}_{p^h}$ of the incidence matrix of points and d-flats in* $\mathrm{PG}(r, p^h)$ *is equal to*

$$R_d(r, p^h) = \sum_{s_0} \cdots \sum_{s_{h-1}} \prod_{j=0}^{h-1} \sum_{i=0}^{L(s_{j+1},s_j)} (-1)^i \binom{r+1}{i} \binom{r + s_{j+1}p - s_j - ip}{i},$$

where $s_h = s_0$, the summations are taken over all integers s_j, $j = 0, \ldots, h-1$, such that $d+1 \le s_j \le r+1$, $0 \le s_{j+1}p - s_j \le (r+1)(p+1)$, and $L(s_{j+1}, s_j)$ is the greatest integer not exceeding $(s_{j+1}p - s_j)/p$, i.e.

$$L(s_{j+1}, s_j) = \lfloor \frac{s_{j+1}p - s_j}{r} \rfloor.$$

Let us note that in Hamada's notation $A = R_1(r, p)$. The formula above is not very handy. For the special case of $d = 1$, i.e. a points-by-lines incidence matrix, and $h = 1$, i.e. a prime field, we have a closed formula for the rank found by van Lint. It is stated below as a corollary and can be found in [2].

Corollary 2. *For the points-by-lines incidence matrix of* $\mathrm{PG}(r, q)$ *we have*

$$\mathrm{rk}_p\, R_1(r, p) = \frac{p^{r+1}-1}{p-1} - \binom{p+r-1}{r}.$$

Corollary 3. *The dimension of the vector space of all $(0 \mod p)$-arcs is* $\dim V = \binom{p+r-1}{r}$.

Now we can characterize the vector space V of all $(0 \mod p)$-arcs.

Theorem 7. *[11] The vector space of all $(0 \mod p)$-arcs in $\mathrm{PG}(r, p)$ is generated by the complements of the hyperplanes.*

Now since the arc associated with $\boldsymbol{j} - \chi_T$ is lifted from every point of the hyperplane T, we have the following corollary.

Corollary 4. *Every* $(0 \mod p)$*-arc in* $\mathrm{PG}(r,p)$ *is a sum of lifted arcs.*

Corollary 5. *Every* $(t \mod p)$*-arc in* $\mathrm{PG}(r,p)$ *is a sum of lifted arcs.*

In the plane case we can prove even more. A $(t \mod p)$-arc can be represented as the sum of at most p lifted arcs.The proof relies on a result by Blokhuis and Moorhouse on the p-rank of certain incidence matrices obtained from ovals in $\mathrm{PG}(2,p)$ [1].

Theorem 8. *[11] Let* $P_1, \ldots, P_p$ *be* p *points from a conic in* $\mathrm{PG}(2,p)$*. Denote by* V_i *the vector space of all* $(0 \mod p)$*-arcs lifted from* P_i*,* $i = 1, \ldots, p$*, and by* V *the vector space of all* $(0 \mod p)$*-arcs. Then*

$$V = V_1 + V_2 + \cdots + V_p.$$

Corollary 6. *Every* $(t \mod p)$*-arc in* $\mathrm{PG}(2,p)$ *can be represented as the sum of at most* p *lifted arcs.*

Recall that the $(t \mod q)$-arcs obtained from t-quasidivisible arcs have the additional property that the point multiplicities are upperbounded by t.

In the plane case, non-trivial $(t \mod q)$-arcs can be constructed as σ-duals of certain blocking sets. Let $\mathcal{K}$ be a multiset in Σ. Consider a function σ such that $\sigma(\mathcal{K}(H))$ is a non-negative integer for all hyperplanes H. The multiset

$$\widetilde{\mathcal{K}}^\sigma : \begin{cases} \mathcal{H} \to & \mathbb{N}_0 \\ H \mapsto \sigma(\mathcal{K}(H)) \end{cases} \tag{3}$$

in the dual space $\widetilde{\Sigma}$ is called the σ*-dual* of $\mathcal{K}$. If σ is a linear function, the parameters of $\widetilde{\mathcal{K}}^\sigma$, as well as its spectrum, are easily computed from the parameters and the spectrum of $\mathcal{K}$ [13].

Theorem 9. *[12] A* $(t \mod q)$*-arc in* $\mathrm{PG}(2,q)$ *of size* $mq+t$ *and with maximal point multiplicity* t *exists if and only if there exists a blocking set in* $\mathrm{PG}(2,q)$ *with parameters* $((m-t)q+m, m-t)$ *and with line multiplicities in the set* $\{m-t, m-t+1, \ldots, m\}$*.*

4 (3 mod 5)-arcs in PG$(3, 5)$ and the nonexistence of some Griesmer codes

First, we classify some small (3 mod 5)-arcs in PG$(2, 5)$. Due to Theorem 9, the classification of such arcs is equivalent to the classification of certain blocking sets with an additional restriction on the line multiplicities.

Arcs of cardinality 18

These arcs correspond to (3,0)-blocking sets with lines of multiplicity 0, 1 2, 3. Such blocking sets correspond to three (not necessarily different) points. These correspond to the sum of three not necessarily different lines in various mutual positions. It is an easy check that there exist four (3 mod 5)-arcs of cardinality 18.

Arcs of cardinality 23

These arcs correspond to $(9, 1)$-blocking sets with lines of multiplicity 1, 2, 3, 4. Hence blocking sets containing a full line do not lead to (3 mod 5)-arcs. Thus the only possibility is the projective triangle. Dualizing we get a (3 mod 5-arc in which the 2-points form a complete quadrangle, the intersections of the diagonals are 3-points and the intersections of the diagonals with the sides of the quadrangle are 1-points.

Arcs of cardinality 28

These arcs are obtained from (15,2)-blocking sets with lines of multiplicity 2, 3, 4, or 5. If such a blocking set does not have multiple points it is obtained as the complement of a (16,4)-arc. Such an arc should not have external lines since the maximal multiplicity of a line with respect to the blocking set is 5. The classification of the (16,4)-arcs is well-known. There exists exactly one such arc without external lines obtained by deleting the common points of six lines in general position from the plane. It is also easily checked that a (15,2)-blocking set having points of multiplicity greater than 1 always has a line of multiplicity 6 and hence does not give a (3 mod q)-arc with point multiplicity at most 3.

Arcs of cardinality 33

If $\mathcal{F}$ is such an arc then $\mathcal{F}^\sigma$ is a $(21, 3)$-blocking set with line multiplicities $3, 4, 5, 6$. Again such a blocking set cannot have points of multiplicity 3 or larger since this would impose lines of multiplicity larger than 6 in $\mathcal{F}$.

Denote by Λ_i the number of points of multiplicity i. Constructions are possible only for $\Lambda_2 = 0, 1, 2$. In such case, $\mathcal{F}^\sigma$ is one of the following:

(1) the complements of the seven non-isomorphic $(10, 3)$-arcs; $\Lambda_2 = 0$;

(2) the complement of the $(11, 3)$-arc with four external lines and a double point a point not on an external line, $\Lambda_2 = 1$;

(3) one double point which forms an oval with five of the 0-points; the tangent in the 2-point is a 3-line, $\Lambda_2 = 1$;

(4) $\mathrm{PG}(2, 5)$ minus a triangle with vertices of multiplicity $2, 2, 1$; $\Lambda_2 = 2$.

Based on the classification for small $(3 \mod 5)$-arcs one can prove that $(3 \mod 5)$-arcs in $\mathrm{PG}(3, 5)$ of small cardinality are always lifted. We have the following theorem.

Theorem 10. *Every* $(3 \mod 5)$*-arc* $\mathcal{F}$ *in* $\mathrm{PG}(3, 5)$ *with* $|\mathcal{F}| \leq 168$ *is a lifted arc. In particular, if* $|\mathcal{F}| \leq 168$ *then* $|\mathcal{F}| = 93, 118, 143,$ *or* 168.

This result enables us to rule out the non-existence of some hypothetical Griesmer codes of dimension $k = 4$ over $q = 5$.

Theorem 11. *There exists no* $(204, 42)$*-arc in* $\mathrm{PG}(3, 5)$*. Equivalently, there exists no* $[204, 4, 162]_5$ *code.*

Sketch of proof. Assume $\mathcal{K}$ is a $(204, 42)$-arc in $\mathrm{PG}(3, 5)$. The restriction of $\mathcal{K}$ to a maximal plane is one of the following: (a) two copies of the plane minus three non-concurrent lines minus two points (extendable to $(44, 9)$-arc); (b) twelve 2-points and a 0-point with all lines containing four 2-points passing also through the 0-point; (c) two copies of the plane minus an $(11, 3)$-arc. In particular a maximal plane does not have 0- or 1-lines.

If (a_i) is the spectrum $\mathcal{K}$ then

$$\sum_{i=0}^{42} \binom{42 - i}{2} a_i = -732 + 25\lambda_2, \tag{4}$$

where λ_2 is the number of the 2-points. Using this as well as the structure of the restriction to the maximal planes one can rule out the existence of planes of multiplicity ≤ 28.

Based on the classification of the arcs with parameters $(32, 7)$-, $(37, 8)$ and $(38, 8)$ which is known, we rule out the existence of 32-, 37- and 38-planes. This implies that $\widetilde{\mathcal{K}}$ is a $(3 \mod 5)$-arc with maximal point multiplicity 3. Furthermore, there is no 18-plane (with respect to $\widetilde{\mathcal{K}}$) which is a line of 3-points. This uses the fact that 0-points of $\widetilde{\mathcal{K}}$ correspond to maximal planes of $\mathcal{K}$. By Theorem 10 and by the classification of the small $(3 \mod 5)$-arcs we get $|\widetilde{\mathcal{K}}| > 168$. This fact together with (4) is used then to rule out the possibility of 42-planes of the types (a) and (b). We shall demonstrate this argument on the case of a 42-plane π for which $\mathcal{K}|_{\pi_0}$ is a $(42, 9)$-arc of type (a) with spectrum $b_9 = 18$, $b_8 = 10$, $b_4 = 2$, $b_2 = 1$.

Consider a line L in π_0 and denote by π_i, $i = 1, \ldots, 5$, the other planes through L. The table below gives the maximal contributions of the planes through L to the left-hand side of (4).

$\lvert\widetilde{\mathcal{K}}(L)\rvert$ $\lvert\mathcal{K}(L)\rvert$	3	8	13
9	3	No	No
8	28	7	No
4	No	156	112
2	No	211	187

Denote by x the number of 8-lines L in π_0 with $|\widetilde{\mathcal{K}}(L)| = 3$, by y the number of 4-lines L in π_0 with $|\widetilde{\mathcal{K}}(L)| = 8$, and by z the number of 2-lines L in π_0 with $|\widetilde{\mathcal{K}}(L)| = 8$. From $|\widetilde{\mathcal{K}}| > 168$ we get

$$18 \cdot 3 + 3x + 8 \cdot (10 - x) + 8y + 13 \cdot (2 - y) + 8z + 13 \cdot (1 - z) > 168,$$

which implies $x + y + z < 1$, i.e. $x = y = z = 0$. Now from (4) we get

$$18 \cdot 3 + 10 \cdot 7 + 2 \cdot 112 + 1 \cdot 187 \geq -732 + 25\lambda_2,$$

whence $\lambda_2 \leq 50$. This is a contradiction since π_0 has four 0-points which implies $\lambda_2 \geq 52$. The other $(42, 9)$-arcs of type (a) and (b) are ruled out in a similar fashion.

Since a (205,42)-arc does not exist every point is contained in a maximal plane. Hence $\mathcal{K}$ does not have 0-points. This implies that the 2-points form a set of 48 points no four of which a colinear. This is impossible since the maximal size of a $(n, 3)$-cap in PG$(3, 5)$ is 43 [4]. $\square$

This leaves just two undecided cases for the optimal length of a four-dimensional code over $\mathbb{F}_5$: $d = 81$ and $d = 161$.

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