Homogeneous Arcs and Linear Codes over Finite Chain Rings

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Abstract. We introduce and study a new class of arcs in the projective geometries over finite chain rings. We call such arcs homogeneous and point out their connection with the linear codes with homogeneous weight. We characterize constant weight homogeneous arcs as sums of neighbor classes of points. We prove some necessary conditions for the existence of two-weight homogeneous arcs and list all known examples for such arcs.

Keywords: projective Hjelmslev geometries, finite chain rings, homogeneous arcs, homogeneous weight, two-weight codes, two-weight arcs, regular arcs

1 Introduction

In this paper we consider finite chain rings R of length 2 with residue field $R/\mathrm{Rad}R$ isomorphic to $\mathbb{F}_q$. It is known that for such rings $|R| = q^2$, and $|\mathrm{Rad}R| = q$, where $q = p^h$ for some prime p. Moreover, there exist exactly $h+1$ isomorphism classes for such rings:

- the factor-rings of skew-polynomial rings in one variable $\mathbb{F}_q[X, \sigma]/(X^2)$, where σ is an automorphism of $\mathbb{F}_q$; these will be denoted by $\mathbb{S}_q^\sigma$, and

- the Galois ring $\mathrm{GR}(q^2, p^2)$, denoted by $\mathbb{G}_q$.

A more in-depth introduction into the structure and basic properties of finite chain rings the reader is referred to [18,19,20].

The (right) $(k-1)$-dimensional Hjelmslev geometry $\mathrm{PHG}(k-1, R)$ (or just Σ_{k-1} if the ring is clear from the context) is defined as an incidence structure in which points are the free submodules of R_R^k of rank 1, lines are the free submodules of R_R^k of rank 2 and incidence is given by inclusion. The free submodules of R_R^k are called Hjelmslev subspaces of the geometry while the non-free submodules are called subspaces. In particular, hyperplanes are the free submodules of rank $k-1$. In what follows, we denote by $\mathcal{P}$ (resp. $\mathcal{H}$) the set of all points (resp. all hyperplanes) of Σ_{k-1}. Two points x, y are neighbors (notation $x \frown y$) if there exist at least two lines incident with both of them. Two Hjelmlsev subspaces S and T of the same dimension are neighbors if every point of S has a neighbor on T and, conversely, every point of T has a neighbor on S. For a point $x \in \mathcal{P}$, we denote by $[x]$ the set of all points that are neighbors to x. Similarly, for an Hjelmslev subspace S we denote by $[S]$ the set of all Hjemslev subspaces that are neighbors of S. The factor structure $\Sigma_{k-1}(R)/\frown$ having as points, lines, planes etc. the neighbor classes on points, lines, planes in $\Sigma_{k-1}(R)$, respectively, is isomorphic to $\mathrm{PG}(k-1, q)$. For more facts about projective Hjelmslev geometries as well as for counting formulae in such geometries, we refer to [8,11,13].

It is known that linear codes over finite chain rings for which the entries in no coordinate position are entirely contained in $\mathrm{Rad}R$ are equivalent to multisets of points (arcs) in $\mathrm{PHG}(k-1, q)$. It turns out however that the Hamming weight does not describe fully the error-correcting properties of the ring codes. The minimum Hamming distance of a linear code over a finite chain ring R is equal to the minimum distance of a rather small subcode – the radical of the code. It is more appropriate to consider linear codes over chain rings with respect to the homogeneous metric introduced by Heise and Constantinescu [4,14]. Switching to the geometric representation of the linear codes, we need a new criterion for the goodness of an arc. In this paper, we introduce and study a new class of arcs which we call homogeneous arcs and point out their connection with linear codes with homogeneous weight. We characterize constant weight homogeneous arcs and list all known examples of two-weight homogeneous arcs.

The extended abstract is structured as follows. In section 2, we introduce homogeneous arcs and point out the connection with the linearly represntable codes over a q-ary alphabet. In section 3 we define the τ-dual of a multiset and give a formula for the homogeneous weights of the τ-dual to a given arc. Section 4 contains a characterization of the arcs with one homogeneous weight which is analogue of a well-known result by Bonisoli about one-weight linear codes and arcs in $\mathrm{PG}(k, q)$ with one intersection number. In section 6 we consider two-weight homogeneous arcs and prove that every such arc is necessarily regular. We describe all known arcs with two homogeneous weights.

2 Homogeneous arcs and linearly representable codes

From now on R will denote a finite chain ring of length 2 with residue field of order q. Let Σ_{k-1} be the (right) projective $(k-1)$-dimensional Hjelmslev geometry over R. Every mapping $\mathcal{K} : \mathcal{P} \to \mathbb{N}_0$ from the pointset $\mathcal{P}$ of Σ_{k-1} to the non-negative integers is called a multiset in Σ_{k-1}. This definition as extended to the power set of $\mathcal{P}$ by

$$\mathcal{K} : \begin{cases} 2^{\mathcal{P}} \to \mathbb{N}_0 \\ S \quad \to \sum_{x \in S} \mathcal{K}(x), \end{cases},$$

where $S \subseteq \mathcal{P}$. Traditionally multisets are viewed as arcs or blocking sets (minihypers) depending on whether we put an upper or a lower bound on the cardinality $\mathcal{K}(H)$ of a hyperplane of Σ_{k-1}. We define the homogeneous weight $\omega_{\mathcal{K}}$ of a subspace S in Σ_{k-1} by

$$\omega_{\mathcal{K}}(S) = \mathcal{K}(S) - \frac{1}{q-1}\mathcal{K}([S] \setminus S), \tag{1}$$

where $[S]$ is the set of all points that are neighbors to the points in S.

A multiset $\mathcal{K}$ is called a homogeneous (N, W)-arc if $\mathcal{K}(\mathcal{P}) = N$, $\omega_{\mathcal{K}}(H) \leq W$ for every hyperplane H, and there exists a hyperplane H_0 with $\omega_{\mathcal{K}}(H_0) = W$. An arc is called projective if the multiplicity of every point is 0 or 1. As in the classical case, the parameters of the complement are easily computed.

Theorem 1. *Let $\mathcal{K}$ be a multiset in Σ_k with homogeneous weights $W_1 < W_2 < \ldots < W_s$ and maximal point multiplicity t, $t = \max_{x \in \mathcal{P}} \mathcal{K}(x)$. Then the arc $\mathcal{K}' = t\mathcal{P} \setminus \mathcal{K}$ has weights $-W_s < \ldots < -W_2 < -W_1$.*

A (left) linear code of length n over the chain ring R is defined as a submodule of ${}_RR^n$. The shape of the submodule is referred to as the shape of the code. In [7,8], a general mapping is defined that transforms every R-linear code into a code (not necessarily linear) over the residue field $\mathbb{F}_q$. In the special case when R is a ring of length 2 this mapping becomes the Reed-Solomon map:

$$\psi_{\mathrm{RS}} \colon \begin{cases} \qquad R \qquad \to \mathbb{F}_q^q, \\ r = r_0 + r_1\theta \to (r_0, r_1)A, \end{cases}$$

where

$$A = \begin{pmatrix} 0 \; 1 \; \zeta \ldots \zeta^{q-2} \\ 1 \; 1 \; 1 \ldots 1 \end{pmatrix},$$

θ is a fixed generator of $\mathrm{Rad}R$, ζ is a primitive element of $\mathbb{F}_q$ and r_i are chosen from and arbitrarily fixed set $\Gamma = \{\gamma_i \mid i = 0, \ldots, q-1\}$ of q elements from

R with $\gamma_i \not\equiv \gamma_j \pmod{\mathrm{Rad}R}$. It is well-known that ψ_{RS} is an isometry from $(R, \mathrm{d}_{\mathrm{hom}})$ into $(\mathbb{F}_q^q, \mathrm{d}_{\mathrm{Ham}})$. Here

$$\mathrm{d}_{\mathrm{hom}}(x,y) = \begin{cases} 0 & \text{if } x = y, \\ q & \text{if } x - y \in \mathrm{Rad}R, \\ q-1 & \text{if } x - y \in R \setminus \mathrm{Rad}R, \end{cases}$$

and $\mathrm{d}_{\mathrm{Ham}}$ is the Hamming distance. The Reed-Solomon map can be extended to the n-tuples over R by

$$\psi_{\mathrm{RS}} : \begin{cases} R^n \to \mathbb{F}_q^{qn} \\ (x_1, x_2, \ldots, x_n) \to (\psi_{\mathrm{RS}}(x_1), \psi_{\mathrm{RS}}(x_2), \ldots, \psi_{\mathrm{RS}}(x_n)) \end{cases}.$$

Let R be a chain ring of length 2 with residue field of order q, $R/\mathrm{Rad}R \cong \mathbb{F}_q$. A code over the alphabet $\mathbb{F}_q$ is said to be linearly representable over the ring R if it is the image of a R-linear code under the Reed-Solomon map. It is known from [7,8] that there is a one-to-one correspondence between the classes of semi-linearly isomorphic left linear codes of full length over R and the classes of projectively equivalent multisets in the (right) geometry $\mathrm{PHG}(R_R^k)$. This correspondence can be used to describe the connection between the parameters of the linearly representable codes over $\mathbb{F}_q$ on one side, and the homogeneous arcs in $\mathrm{PHG}(R_R^k)$, on the other.

Theorem 2. *Let R be a finite chain ring of length 2 with residue field of order q. There exists a correspondence between the set of linearly representable q-ary codes of full length over R with parameters $(Nq, q^{2k_1+k_2}, (q-1)(N-W))$ a homogeneous (N,W)-arcs in $\Sigma_{k-1} = PHG(R_R^k)$, whose support generates a subspace of Σ_{k-1} of shape (k_1, k_2).*

By this theorem the construction of good q-ary linearly representable codes is equivalent to the construction of good arcs with respect to the homogeneous weight.

3 Duality for homogeneous arcs

Let $\Sigma_{k-1} = \mathrm{PHG}(R_R^k)$ and let $\mathcal{P}$ denote its set of points. Consider a multiset $\mathcal{K}$ in Σ_{k-1}. The type of a hyperplane H in Σ_{k-1} is defined as the triple $\mathbf{a}(H) = (a_0(H), a_1(H), a_2(H))$, where $a_0(H) = \mathcal{K}(P) - \mathcal{K}([H])$, $a_1(H) = \mathcal{K}([H]) - \mathcal{K}(H)$, $a_2(H) = \mathcal{K}(H)$. Clearly, $\omega_{\mathcal{K}}(H) = a_2(H) - \dfrac{1}{q-1} a_1(H)$. Denote by $\mathcal{W}_{\mathcal{K}}$ the set of all intersection types of hyperplanes with respect to $\mathcal{K}$: $\mathcal{W}_{\mathcal{K}} = \{\mathbf{a}(H) \mid H \in \mathcal{H}\}$. Following [12] we can define an arc in the dual plane by assigning equal multiplicities to hyperplanes of the same intersection type. In other words, given a function $\tau : \mathcal{W}_{\mathcal{K}} \to \mathbb{N}_0$ we define the τ-dual of $\mathcal{K}$ by

$$\mathcal{K}^\tau : \begin{cases} \mathcal{H} \to \mathbb{N}_0 \\ H \to \tau(\mathbf{a}(H)) \end{cases} .$$

If $\tau(\mathbf{a})$ is linear in the components of $\mathbf{a}$, i.e $\tau(\mathbf{a}) = \alpha + \beta a_1 + \gamma a_2$, $\alpha, \beta, \gamma \in \mathbb{Q}$, we can compute the types of the hyperplanes in the dual geometry with respect to $\mathcal{K}^\tau$ [12].

Theorem 3. *Let $\mathcal{K}$ be a multiset in $PHG(R^k_R)$, where R is a chain ring with $|R| = q^2$, $R/RadR \cong \mathbb{F}_q$. Let $\alpha, \beta, \gamma \in \mathbb{Q}$ be such that $\alpha + \beta a_1 + \gamma a_2 \in \mathbb{N}_0$ for all $\mathbf{a} = (a_0, a_1, a_2) \in \mathcal{W}_{\mathcal{K}}$. For any hyperplane H of type $\mathbf{a} = (a_0, a_1, a_2)$, let*

$$\tau(H) = \tau(\mathbf{a}(H)) = \alpha + \beta a_1 + \gamma a_2.$$

Then the type of an arbitrary hyperplane $x^* = \mathbf{x}R \in \mathcal{P}$ in the dual geometry is $\mathbf{b} = (b_0, b_1, b_2)$, where

$$\begin{aligned}
b_0 &= \alpha q^{2k-2} + \beta n q^{2k-4}(q-1) + \gamma n q^{2k-4} \\
&\quad - \Big(\beta q^{2k-4}(q-1) + \gamma q^{2k-4}\Big)\mathcal{K}([x]), \\
b_1 &= \alpha q^{k-2}(q^{k-1}-1) + \beta n q^{k-3}(q^{k-2}-1)(q-1) + \gamma n q^{k-3}(q^{k-2}-1) \\
&\quad + \Big(\beta q^{k-3}(q^k - 2q^{k-1} + q^{k-2} - 1) + \gamma q^{k-3}(q^{k-1} - q^{k-2} + 1)\Big)\mathcal{K}([x]) \\
&\quad - (\gamma - \beta) q^{2k-4}\mathcal{K}(x), \\
b_2 &= \alpha q^{k-2} \cdot \frac{q^{k-1}-1}{q-1} + \beta n q^{k-3}(q^{k-2}-1) + \gamma n q^{k-3} \cdot \frac{q^{k-2}-1}{q-1} \\
&\quad + \Big(\beta q^{k-3}(q^{k-1} - q^{k-2} + 1) + \gamma q^{k-3}(q^{k-2}-1)\Big)\mathcal{K}([x]) \\
&\quad + (\gamma - \beta) q^{2k-4}\mathcal{K}(x).
\end{aligned}$$

This theorem enables us to compute the homogeneous weight of a hyperplane x^*, $x \in \mathcal{P}$, in the dual geometry Σ^*_{k-1}.

Corollary 1. *Let R, $\mathcal{K}$ and τ be as in Theorem 3. Then the homogeneous weight of the hyperplane x^* with respect to the dual arc $\mathcal{K}^\tau$ is given by*

$$\omega_{\mathcal{K}^\tau}(x^*) = \frac{(\gamma - \beta) q^{k-2}}{q-1} (q^{k-1}\mathcal{K}(x) - \mathcal{K}([x])).$$

4 Constant weight homogeneous arcs

It is known that a linear code in which all non-zero words assume the same weight is a direct sum of simplex codes [1]. For arcs in $\mathrm{PG}(k-1, q)$ this means that if all hyperplanes have the same multiplicity with respect to some arc then it is the sum of several copies of the whole projective space. The theorem below is an analogue of this result for arcs with constant homogeneous weight.

Lemma 1. *Let $\mathcal{K}$ be an (N, W)-homogeneous arc in Σ_k. For an arbitrary hyperplane H_0*

$$\sum_{H \in [H_0]} \omega_{\mathcal{K}}(H) = 0,$$

where the sum is over all hyperplanes of Σ_{k-1} that are neighbors to H. In particular, if $\mathcal{K}$ is an arc in $\Sigma_{k-1}(R)$ for which all hyperplanes have constant homogeneous weight W then $W = 0$.

Theorem 4. *Every $(N, 0)$-homogeneous arc is a sum of neighbor classes of points.*

Proof. Order linearly the points x_i and the hyperplanes H_i, in such way that $x_i \frown x_j$ (resp. $H_i \frown H_j$) iff $\lfloor i/q \rfloor = \lfloor j/q \rfloor$. Here i runs the integers $0, 1, \dots, q^{k-1}\frac{q^k-1}{q-1} - 1$. For this linear ordering of points and hyperplanes, define the square matrix $A = (a_{ij})$ of size $q^{k-1}\frac{q^k-1}{q-1}$ by

$$a_{ij} = \begin{cases} 1 & \text{if } x_i \in H_j, \\ -\frac{1}{q-1} & \text{if } x_i \notin H_j \text{ but } x_i \frown H_j, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

With every homogeneous arc $\mathcal{K}$ we associate a vector

$$\mathbf{x}_{\mathcal{K}} = \left(\mathcal{K}(x_0), \mathcal{K}(x_1) \dots, \mathcal{K}(x_{q^{k-1}\frac{q^k-1}{q-1}-1}) \right).$$

If $\mathcal{K}$ is a $(N, 0)$ arc then $\mathbf{x}_{\mathcal{K}} A = \mathbf{0}$, where $\mathbf{0} = (0, 0, \dots, 0)$ is of length $q^{k-1}\frac{q^k-1}{q-1}$.

Now we are going to prove that $\mathrm{rk} A = (q^{k-1} - 1)\frac{q^k-1}{q-1}$. Consider the matrix $A' = (a|B)$, where B is a $q^{k-1}\frac{q^k-1}{q-1}$ by $\frac{q^k-1}{q-1}$ matrix whose columns are the incidence vectors of the neighbor classes of points:

$$b_{ij} = \begin{cases} 1 & \text{if } x_i \in [x_{jq^{k-1}}] \\ 0 & \text{if } x_i \notin [x_{jq^{k-1}}]. \end{cases}$$

The characteristic vectors $\chi(H_i)$ of the hyperplanes of Π_k belong to the vector space spanned by the columns of A'. Since the incidence matrix of all s-dimensional versus all t-dimensional Hjelmslev subspaces is of full rank, the matrix A' is of full rank [17], i.e.

$$\mathrm{rk} A' = q^{k-1}\frac{q^k - 1}{q - 1}.$$

This implies that

$$\mathrm{rk}A \geq q^{k-1}\frac{q^k-1}{q-1} - \frac{q^k-1}{q-1} = (q^{k-1}-1)\frac{q^k-1}{q-1}.$$

On the other hand, we have

$$\sum_{j:\lfloor j/q^{k-1}=a\rfloor} A^{(j)} = \mathbf{0},$$

for all $a \in \{0,1,\ldots,\frac{q^k-1}{q-1}-1\}$. Here $A^{(j)}$ are the columns of A. This implies that

$$\mathrm{rk}A = (q^{k-1}-1)\frac{q^k-1}{q-1},$$

and the space of all solutions is spanned by the vectors

$$\mathbf{b}_a = (\underbrace{0,0,\ldots,0}_{aq^{k-1}},\underbrace{1,1,\ldots,1}_{q^{k-1}},\underbrace{0,0,\ldots,0}_{q^{k-1}\frac{q^k-1}{q-1}-q-1}),$$

where $a = 0,1,\ldots,\frac{q^k-1}{q-1}-1$. $\square$

By Theorem 4 it suffices to consider only arcs in which every neighbor class of points contains a 0-point.

5 Two-weight homogeneous arcs

An arc with two homogeneous weight is called a two-weight homogeneous arc. If the two weights are W_1 and W_2, we have obviously $W_1 < 0 < W_2$. By Lemma 1 every neighbor class of hyperplanes contains hyperplanes of both homogeneous weights. A two-weight homogeneous arc with homogeneous weights W_1 and W_2 is called a homogeneous arc of type (W_1, W_2). An arc $\mathcal{K}$ is called regular if the multiplicity of every neighbor class of points is constant.

Lemma 2. *Let $\mathcal{K}$ be a regular arc with s different hyperplane multiplicities*

$$U_1 < U_2 < \ldots < U_s.$$

If $\mathcal{K}([x]) = c$ then $\mathcal{K}$ is an arc with s homogeneous weights W_i, $i = 1,\ldots,s$, where $W_i = \dfrac{q}{q-1}U_i - \dfrac{c}{q-1}$.

It follows by this lemma that for regular arcs it makes no difference whether we consider homogeneous or classical arcs. An arc with two homogeneous weights must necessarily be regular.

Theorem 5. *Let $\mathcal{K}$ be an arc in $\Sigma_{k-1}(R)$ with two homogeneous weights. Then $\mathcal{K}$ is regular.*

The existence of a homogeneous arc $\mathcal{K}$ of type (W_1, W_2) in the right geometry $\mathrm{PHG}(k-1, R)$ with $|\mathcal{K}| = N$ is equivalent to that of a left linear code $C <_R R^N$ of full length with two homogeneous weights $w_1 = N - W_1$ and $w_2 = N - W_2$. Hence the projective homogeneous arcs with two weights determine strongly regular graphs with parameters (V, K, λ, μ), where

$$\begin{aligned} V &= |C|, \\ K &= \frac{(N-w_2)|C| + w_2}{w_1 - w_2}, \\ \lambda &= \frac{NK\left(1 - (1 - \frac{w_1}{N})^2\right) + w_2(1-K)}{w_1 - w_2}, \\ \mu &= \frac{NK\left(1 - (1-\frac{w_1}{N})(1-\frac{w_2}{N})\right) - w_2 K}{w_1 - w_2}, \end{aligned}$$

as noted in [2,5,15]. The following theorem was proved in a slightly different form in [5]. It is analogous to a divisibility result by Calderbank and Kantor [3] (Corollary 5.5).

Theorem 6. *Let $\mathcal{K}$ be a two-weight homogeneous arc of type (W_1, W_2) in Σ_{k-1}. Let (x, H) be a point-hyperplane pair with $x \in H$. Then*

(i) $\mathcal{K}([x] \cap H)$ divides q^{k-2};
(ii) for every two hyperplanes $H', H'' \in [H]$, $(q-1)(W_2 - W_1)$ divides

$$q^{k-2}\left(\mathcal{K}([x] \cap H') - \mathcal{K}([x] \cap H'')\right).$$

Below we we present the known classes of homogeneous two-weight arcs.

Example 1. For every chain ring R of length 2 take s, $1 \le s \le q$, points in every neighbor class of points. We have

$$W_1 = -\frac{s}{q-1}, \quad W_2 = \frac{q-s}{q-1}.$$

Example 2. For a chain ring R of length 2 with char$R = p$, i.e. $R = \mathbb{F}_q[X;\sigma]/(X^2)$ take the points of a subgeometry isomorphic to $\mathrm{PG}(2,q)$. Then

$$W_1 = -\frac{1}{p-1}, \quad W_2 = p.$$

Example 3. Take s parallel hyperplane segments with all possible directions in $\mathrm{PHG}(R_R^k)$. Then

$$N = sq^{k-2}\frac{q^k-1}{q-1}, \;\; W_1 = -\frac{sq^{k-2}}{q-1}, \;\; W_2 = q^{k-2}(s - \frac{s-1}{q-1}).$$

In particular, if we set $s = 1$, $k = 3$, and take the points of q^2+q+1 line segments in $\mathrm{PHG}({}_RR^3)$ with all possible directions in the factor plane

$$N = q(q^2+q+1), \;\; W_1 = -\frac{q}{q-1}, \;\; W_2 = q.$$

Example 4. Let $R = \mathbb{F}_q[X;\sigma]/(X^2)$. Take line segments in all possible directions containing the points of $\mathrm{PG}(2,q)$, but delete the points of $\mathrm{PG}(2,q)$. Then

$$N = (q-1)(q^2+q+1), \;\; W_1 = -1, \;\; W_2 = q + \frac{1}{q-1}.$$

Example 5. A hyperoval in $\mathrm{PHG}({}_RR^3)$, where R is a chain ring of length 2 and characteristic 4 (the Galois ring $\mathbb{Z}_4[X]/(f)$), i.e. $q = 2^r$. Then

$$N = q^2+q+1, \;\; W_1 = -\frac{q+1}{q-1}, \;\; W_2 = 1.$$

Example 6. The complement to the dual of a hyperoval in $\mathrm{PHG}({}_RR^3)$, where R is a chain ring of length 2 and characteristic 4, i.e. $q = 2^r$. Then

$$N = \binom{q}{2}(q^2+q+1), \;\; W_1 = -\binom{q+1}{2}, \;\; W_2 = \frac{q}{2(q-1)}.$$

In particular, for $r = 2$, $q = 4$ we get an optimal $(126,8)$-arc with $W_1 = -10$, $W_2 = 2/3$.

Example 7. A sporadic (39,5)-arc in $\mathrm{PHG}(\mathbb{Z}_9^3)$. $W_1 = -2, W_2 = 3/2$ (cf. [6]).

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